Selection Principles and Omission of Intervals

Boaz Tsaban

Bar-Ilan University

TOPOSYM '22

$S_1(A, B)$:

Selecting one member from each family

Measure theory: Strong Measure Zero

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- $\mathsf{S}_1(\mathrm{O}, \mathrm{O})$:

  $\forall\ \mathcal{U}_1, \mathcal{U}_2, \ldots \in \mathrm{O}$, $\exists\ U_1 \in \mathcal{U}_1, U_2 \in \mathcal{U}_2, \ldots$,

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$S_1(A, B)$

$\mathcal{U}_1 \quad \mathcal{U}_2 \quad \mathcal{U}_3 \quad \dots \quad \in \mathcal{A}$

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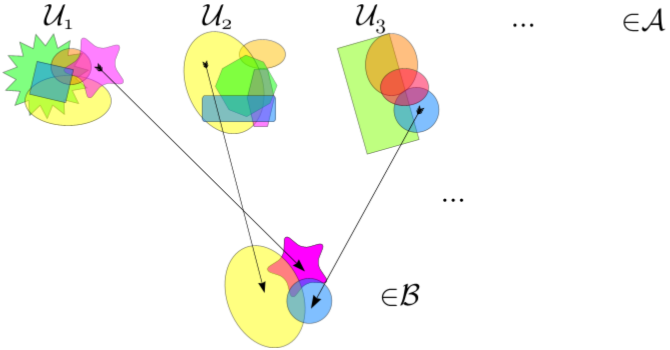

Function spaces and local-to-global duality

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Additional dualities:

$C(X)$ strong fan tightness (Sakai '88) $\iff X\ \mathsf{S}_1(\Omega, \Omega)$

$C(X)$ wQN (Bukovský–Recław–Repický '91) $\iff X\ \mathsf{S}_1(\Gamma, \Gamma)$ (*)

Case study: The Gartside–Lo–Marsh Problem

For X with coarser second countable topology:

$\mathbb{R}^X$ is SSS $\longrightarrow$ $\mathrm{B}(X)$ is SSS $\longrightarrow$ $\mathrm{C}(X)$ is SSS

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Osipov–Szewczak–Ts. '20: Dualizing:

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$\mathsf{S}_{\mathrm{fin}}(\mathrm{A}, \mathrm{B})$:

Selecting finitely many from each family

Dimension theory: Menger's basis property

Hurewicz 1925: A basis property of Menger is equivalent to

$\mathsf{S}_{\mathrm{fin}}(\mathrm{O},\mathrm{O})$: $\forall\, \mathcal{U}_1, \mathcal{U}_2, \dots \in \mathrm{O}$, $\exists$ finite $\mathcal{F}_1 \subseteq \mathcal{U}_1, \mathcal{F}_2 \subseteq \mathcal{U}_2, \dots,$
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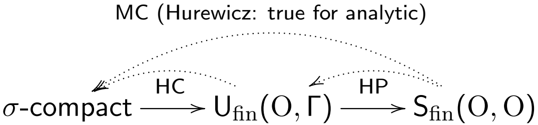

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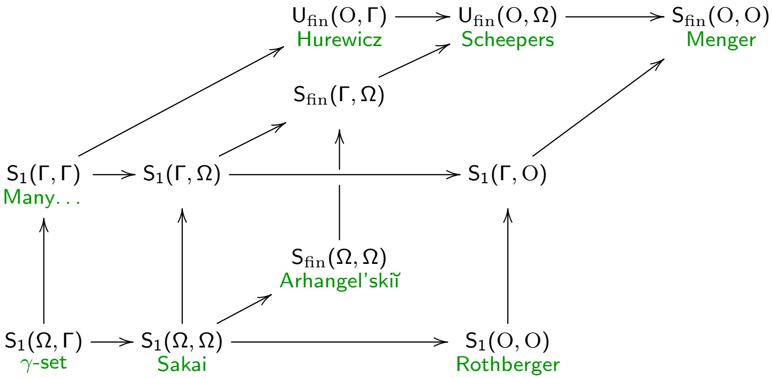

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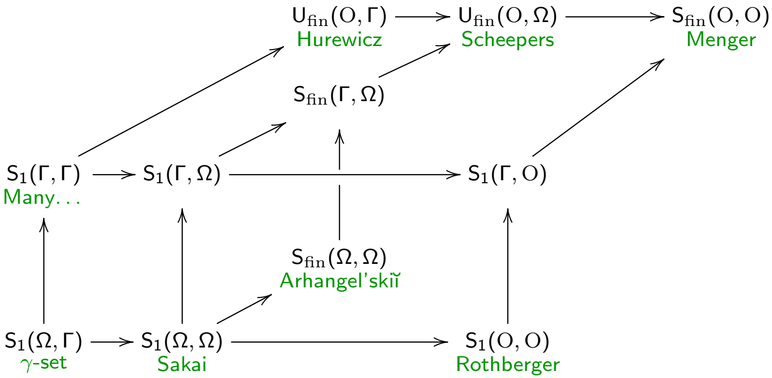
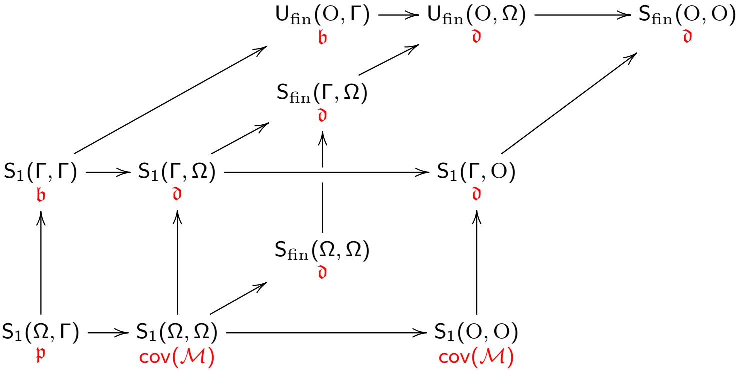
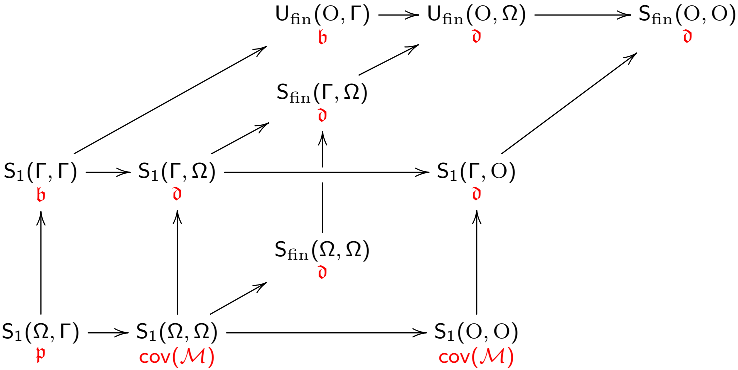


Transferring knowledge, stratification

Combinatorial cardinals





Consistency of inequalities, refined methods, ZFC results (later)

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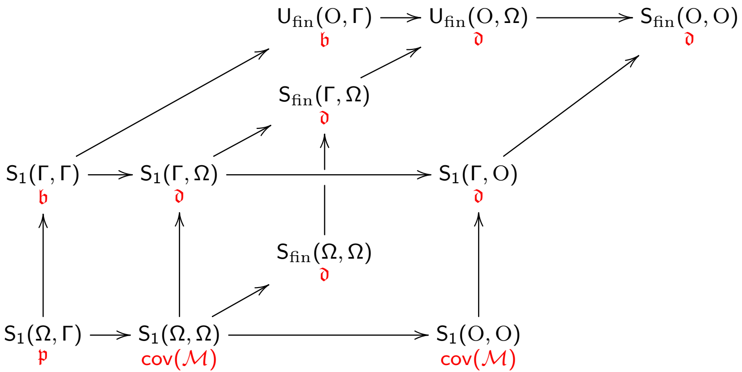

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Paradigm shift: From quantitative to qualitative

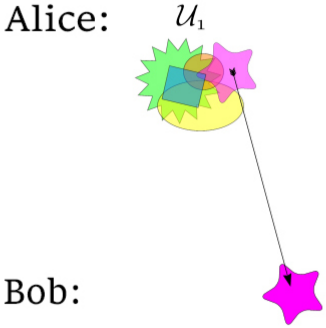
Games and Ramsey theory

The game $G_1(A, B)$

Alice: $\mathcal{U}_1$

Bob:

The game $G_1(A, B)$



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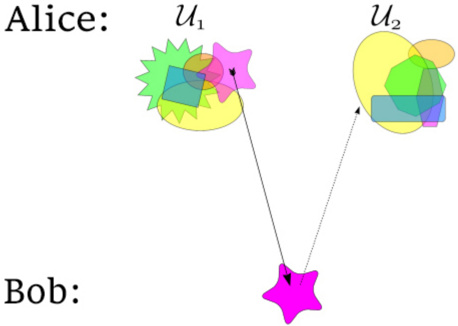

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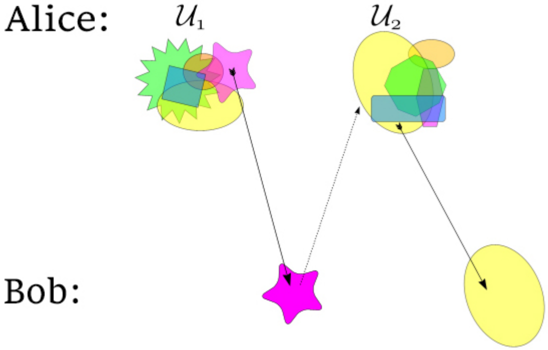

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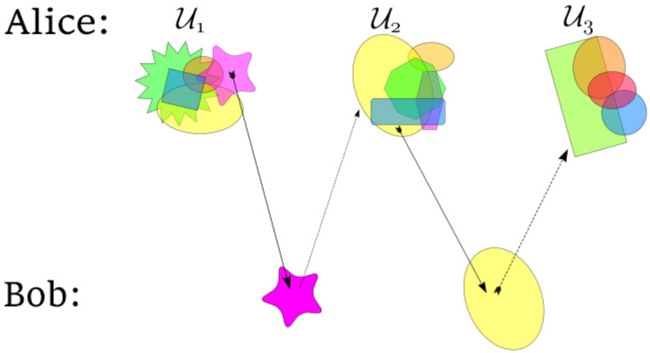

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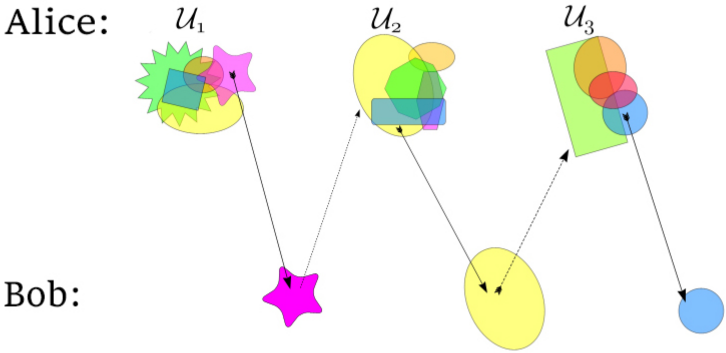

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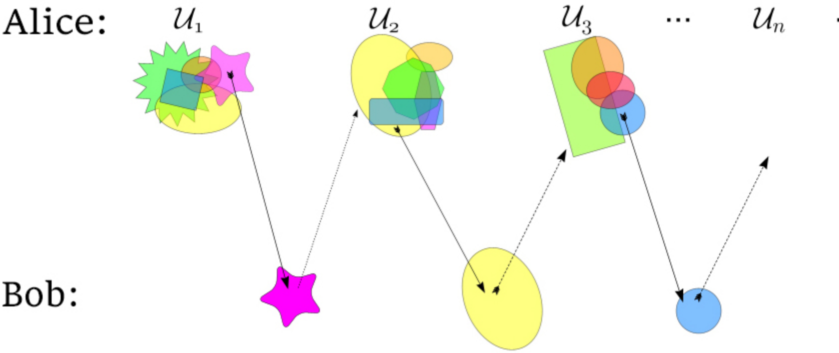

Games and the D-space Problem

Phenomenon (Hurewicz, Pawlikowski, Scheepers, Kočinac,...):

$\mathsf{S}_1(\mathrm{A},\mathrm{B}) \iff$ Alice has no winning strategy in $\mathsf{G}_1(\mathrm{A},\mathrm{B})$

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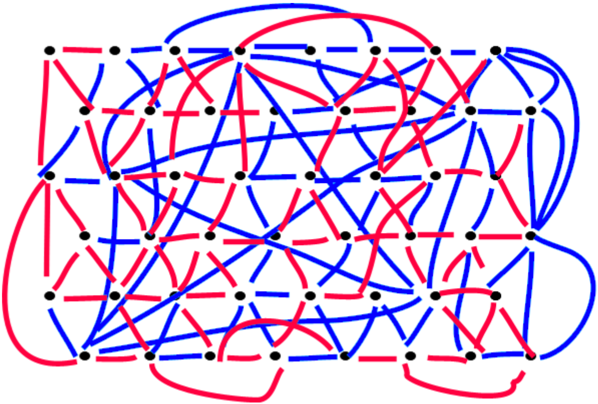
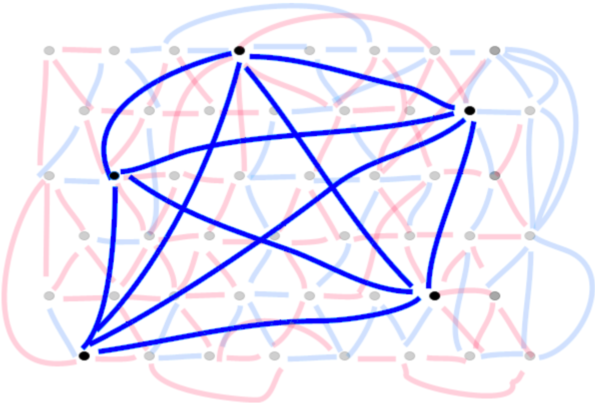
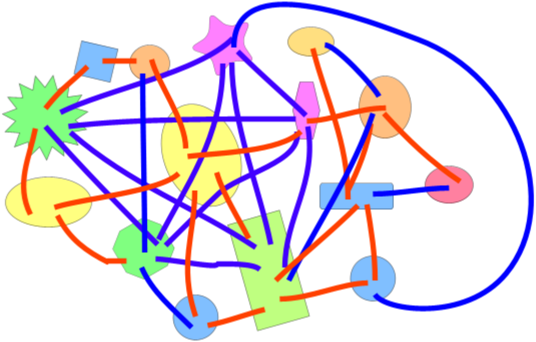
Aurichi '10: Every $\mathsf{S}_{\mathrm{fin}}(\mathrm{O},\mathrm{O})$ space is a "D-space"



Ramsey's Theorem



Ramsey's property for open covers $\mathrm{A} \to (\mathrm{B})^2_2$



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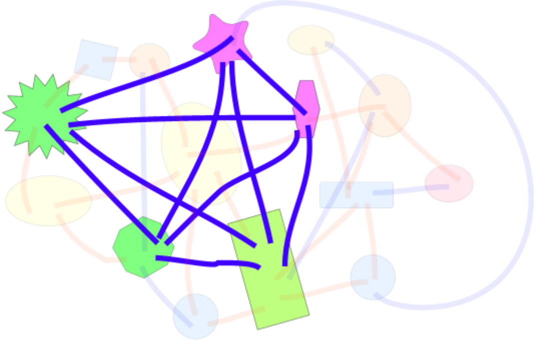

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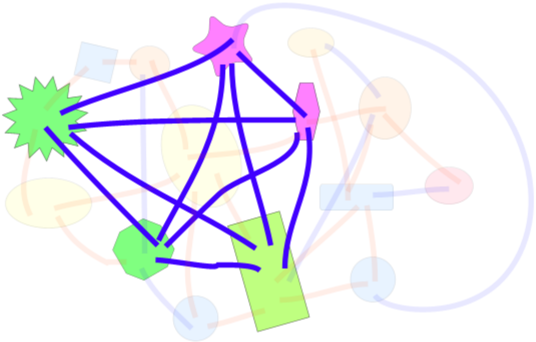

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Phenomenon: $S_1(A, B) \iff A \to (B)^2_2$

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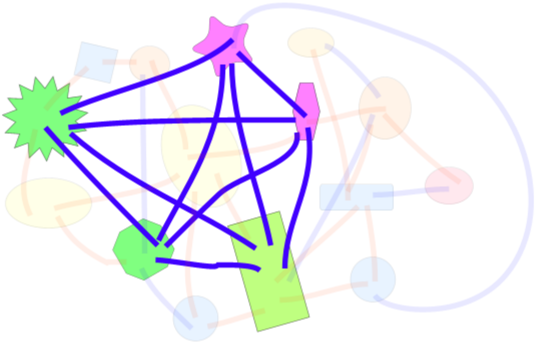

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Most proofs use games

Additive Ramsey theory

Hindman 1974: $\forall$ finite coloring of $\mathbb{N}$, $\exists$ $k_1, k_2, \ldots,$

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Proof uses games

Omission of Intervals

Subsets of the real line

Cantor space $\{0,1\}^{\mathbb{N}} \cong$ Cantor set $\subseteq \mathbb{R}$

$\{0,1\}^{\mathbb{N}} \cong \mathrm{P}(\mathbb{N})$ via characteristic functions

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Build sets of form $S \cup \mathrm{Fin}$, with $S \subseteq [\mathbb{N}]^{\infty}$

The combinatorial structure of S guarantees the selection property

Example: the Hurewicz Problem

Iterated functions: For $y \in [\mathbb{N}]^\infty$ with $1 < y(1)$,

$$\tilde{y}(n) = y^n(1) = \underbrace{y(y(\cdots y(1)\dots))}_{n \text{ times}}$$

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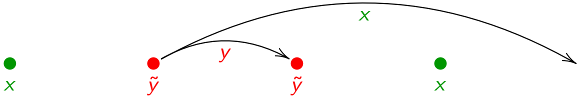

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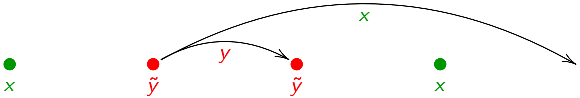

$x := \bigcup_n [\tilde{y}(2n), \tilde{y}(2n+1))$. $y \leq^\infty x, x^c$

Nontrivial Menger non-Hurewicz sets, in ZFC

Lemma: $\forall\ |Y| < \mathfrak{d}$, $a \in [\mathbb{N}]^\infty$,

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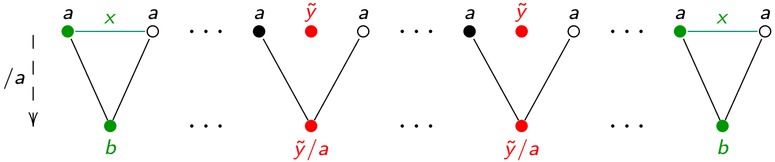

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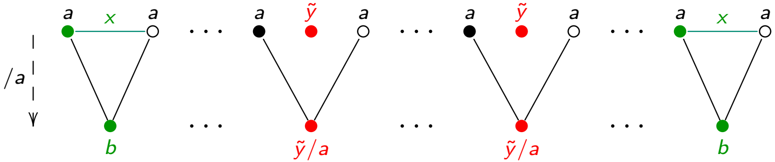

Get

$$\underbrace{\underbrace{\{\, x_\alpha : \alpha < \mathfrak{d} \,\}}_{\mathfrak{d}\text{-unbounded}} \cup \mathrm{Fin}}_{\mathsf{S}_{\mathrm{fin}}(\mathrm{O},\mathrm{O})} \longrightarrow \underbrace{\underbrace{\{\, x_\alpha^c : \alpha < \mathfrak{d} \,\} \cup \mathrm{CoFin}}_{\text{unbounded}}}_{\text{Not } \mathsf{U}_{\mathrm{fin}}(\mathrm{O},\Gamma)} \subseteq [\mathbb{N}]^\infty$$

OMI everywhere

Ts. '11 γ-set Theorem: For every unbounded tower T of height $\mathfrak{p}$, $T \cup \mathrm{Fin}$ is a γ-set $\binom{\Omega}{\Gamma}$

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Szewczak–Wiśniewski '19: Applications to Luzin sets

Szewczak–Włudecka '21 (unb. tower): $\mathsf{S}_1(\Gamma, \Gamma)$ in all finite powers

Szewczak–Ts.–Zdomskyy '21 (regular $\mathfrak{d} \leq \mathfrak{r}$):

- Menger non-Scheepers;
- X, Y $\mathsf{S}_{\mathrm{fin}}(\Omega, \Omega)$, $X \times Y, X \cup Y$ not Menger

Szewczak–Weiss '22 (mild): γ-sets, one null-additive, the other not

The δ-set Problem

$\liminf A_n := \bigcup_k \bigcap_{n\geq k} A_n$

$\{\, U_n \colon n \in \mathbb{N} \,\} \in \Gamma \iff X = \liminf U_n$

L: open covers $\mathcal{U}$ with X in clusure of $\mathcal{U}$ under lim inf

$\Gamma \subseteq L$

δ-set: $\binom{\Omega}{L}$

$\binom{\Omega}{\Gamma} \longrightarrow \binom{\Omega}{L}$

Gerlits–Nagy 1982: $\binom{\Omega}{L} = \binom{\Omega}{\Gamma}$?

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Bardyla–Šupina–Zdomskyy '22 ($\mathfrak{p} = \mathfrak{c}$): $\binom{\Omega}{L} \neq \binom{\Omega}{\Gamma}$!

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Using 2-dimensional omission of intervals

Conclusion

SPs are manywhere: Measure, Dimension, function spaces, . . .

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Omission of Intervals provides a unified framework to find examples with minimal or no assumptions

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THANK YOU!