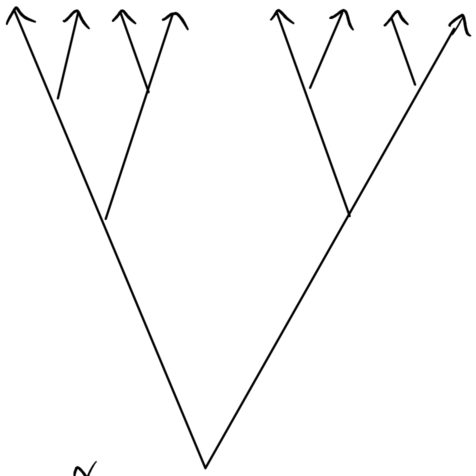
THE TOPOLOGICAL END SPACE PROBLEM

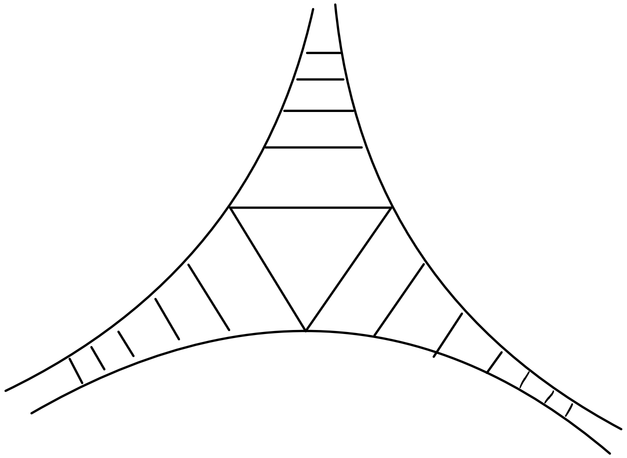
MAX PITZ

TOPOSYM 2022

END SPACES OF GRAPHS



- $2^{\aleph_0}$ many ends



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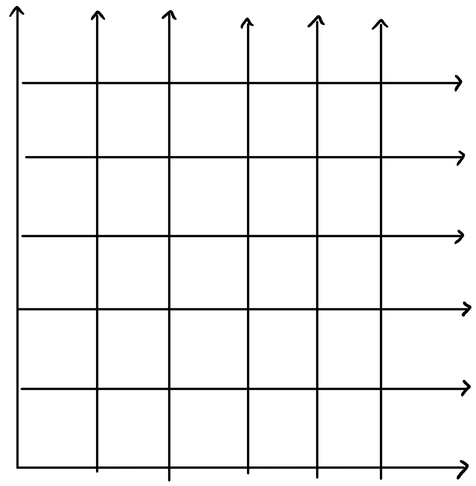

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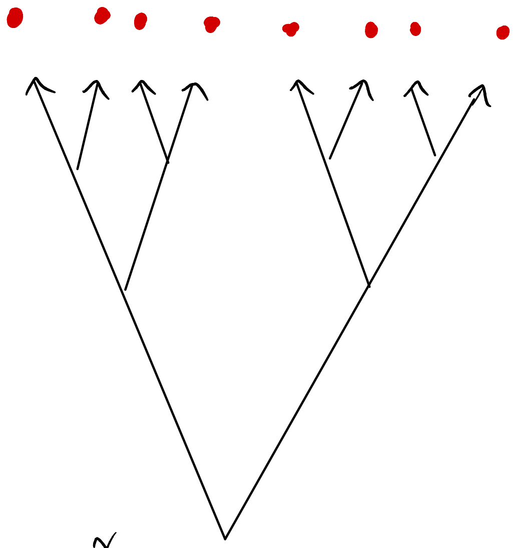

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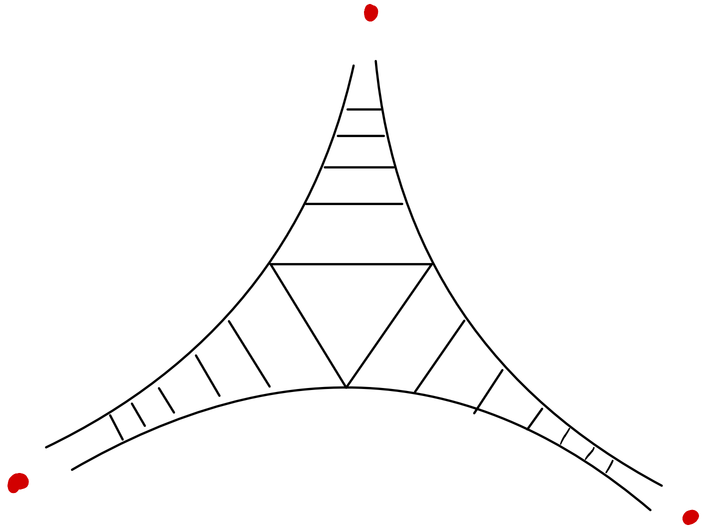

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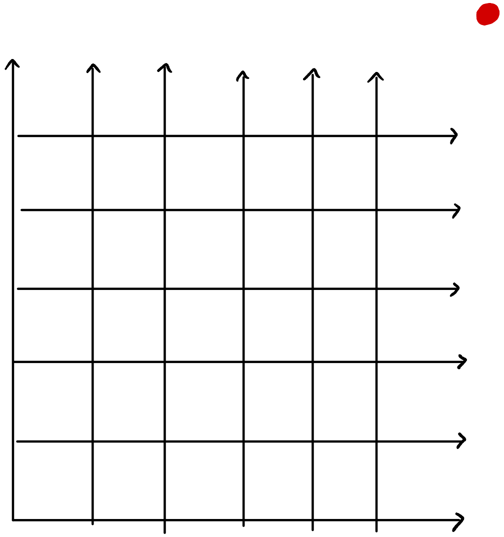

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Intuition: End $\hat{=}$ directions into which graph extends to ∞.

(Freudenthal, Hopf '40's; Halin, Jung '60's)

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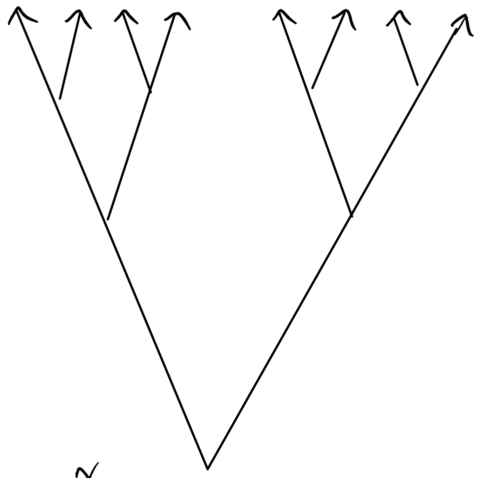

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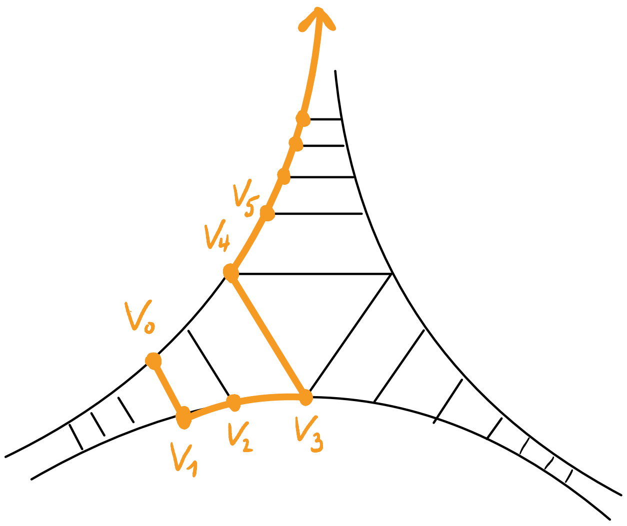

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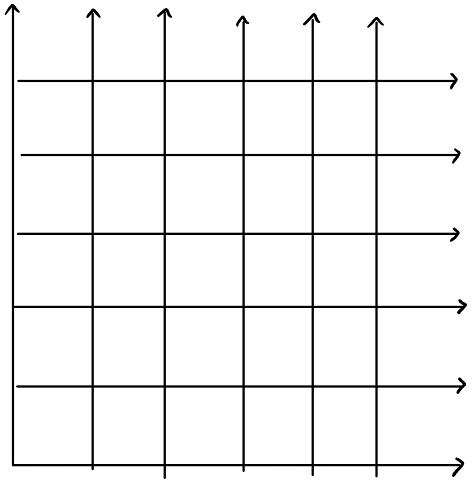

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Def: • Ray = 1-way infinite path $R = v_0 v_1 v_2 v_3 \dots$

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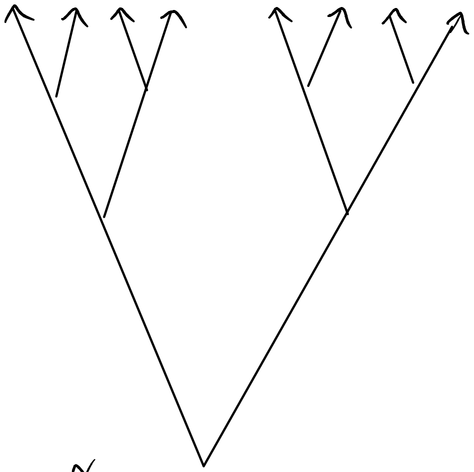

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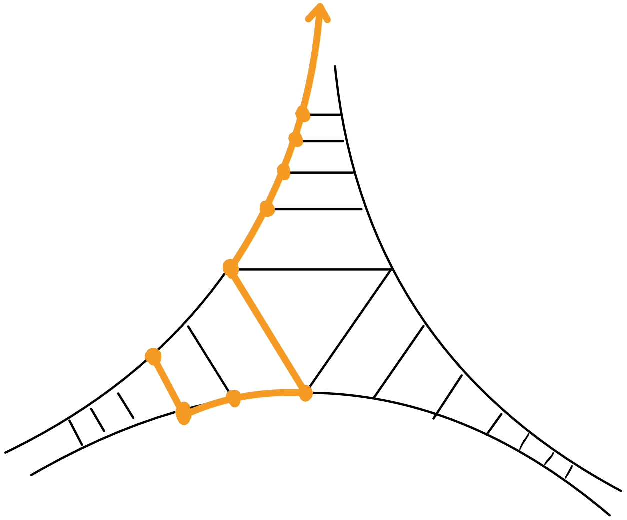

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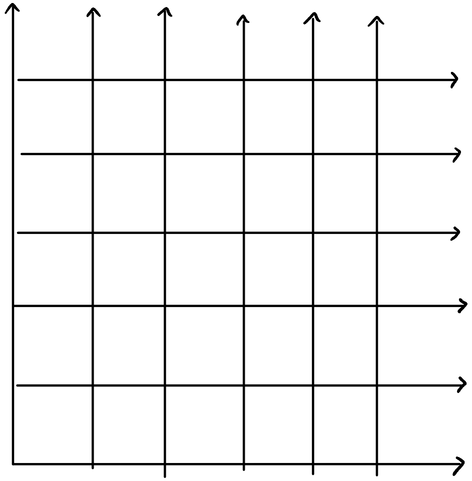

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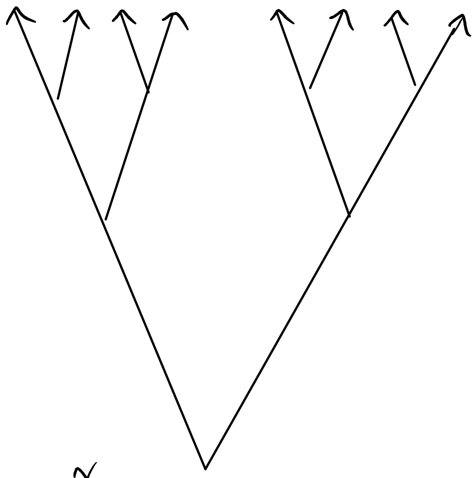

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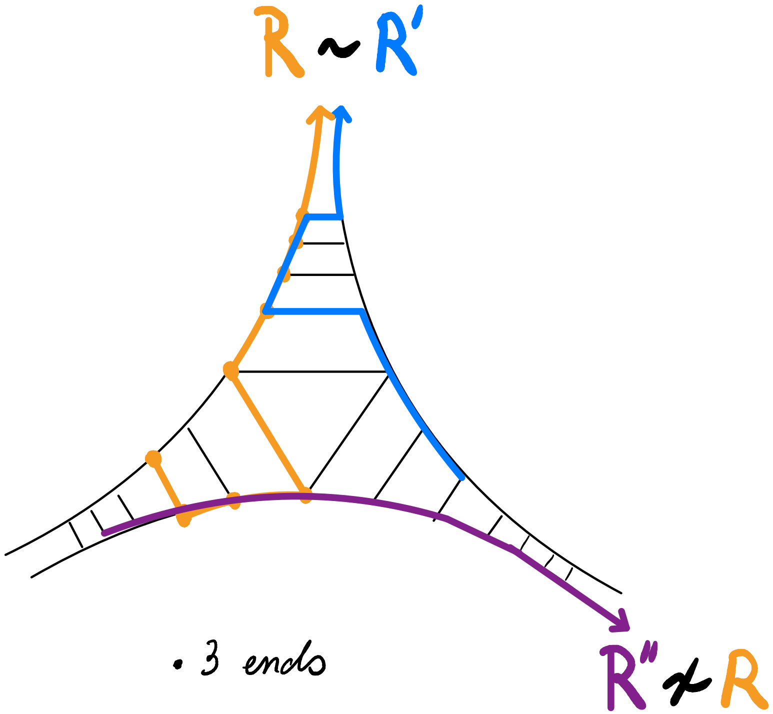

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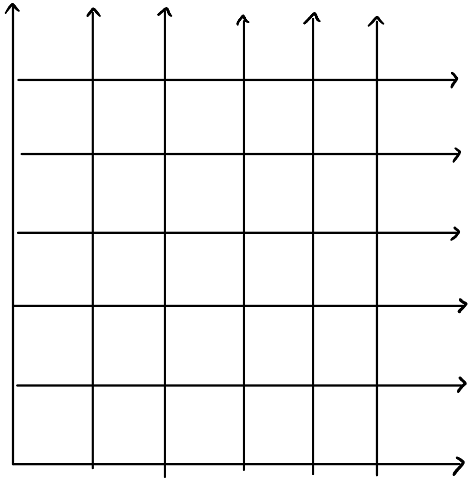

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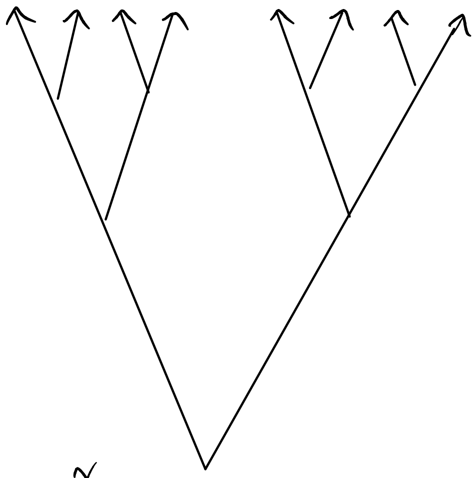

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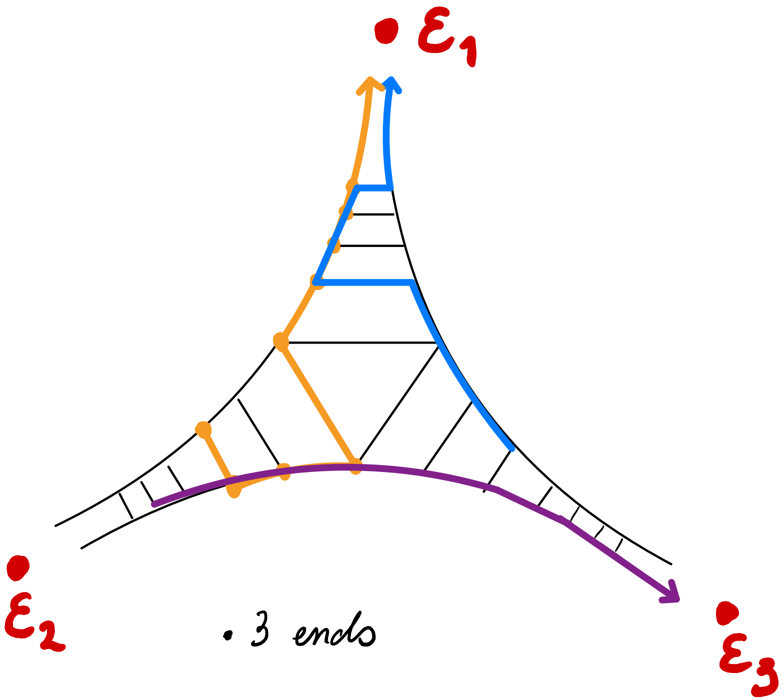

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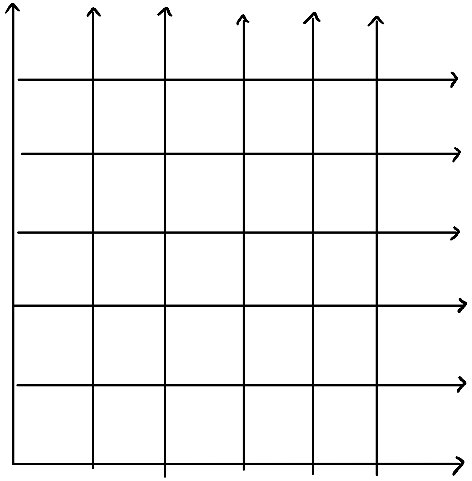

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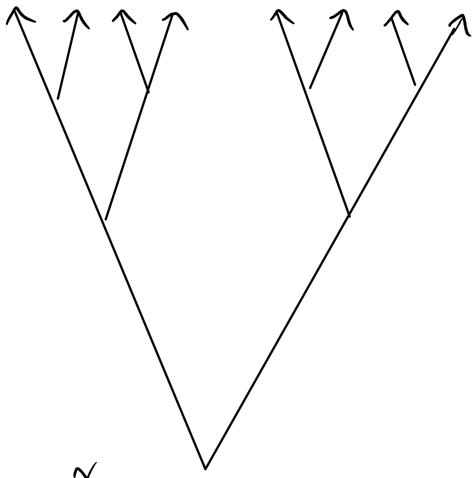

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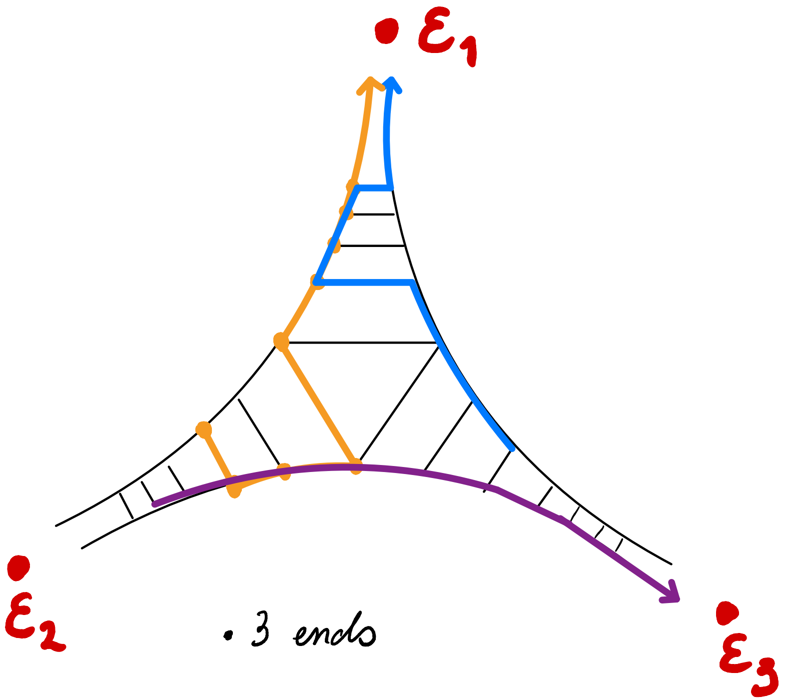

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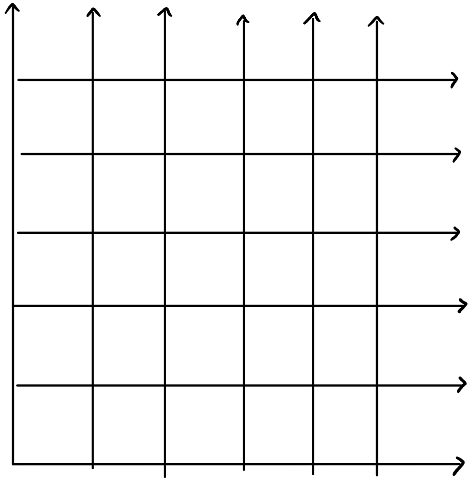

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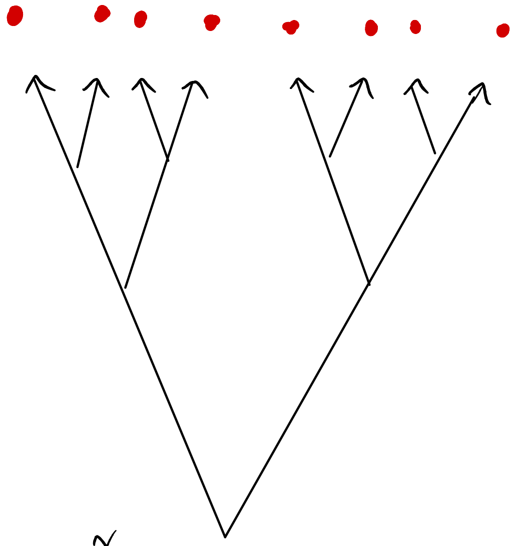

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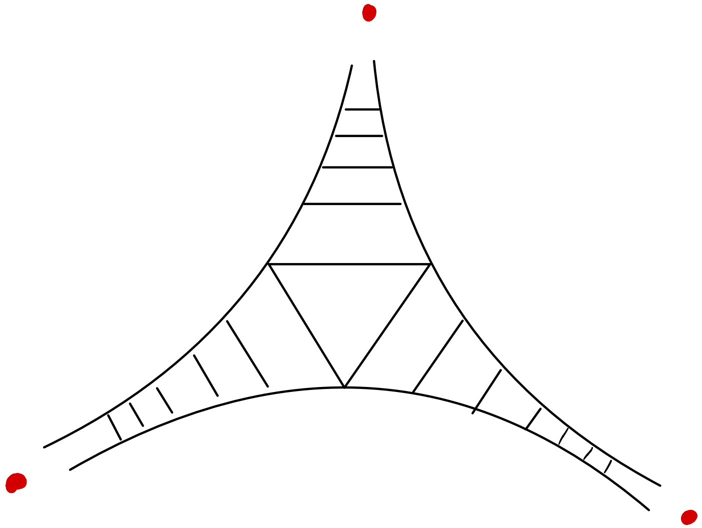

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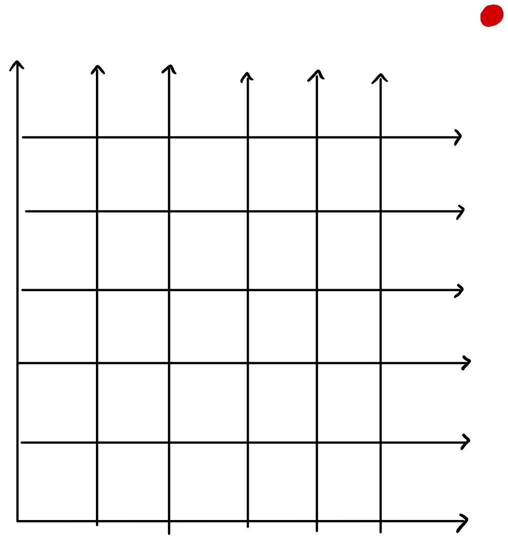

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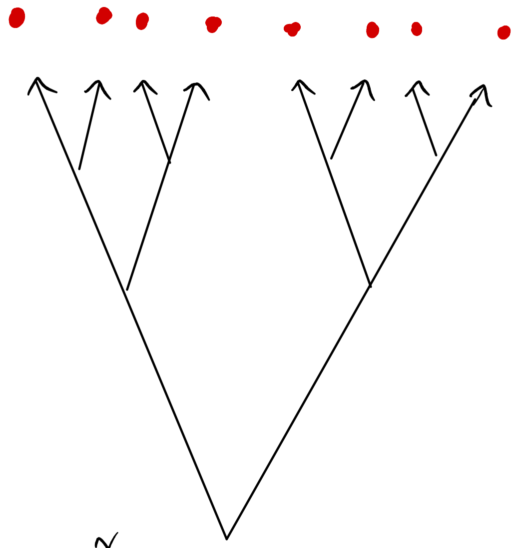

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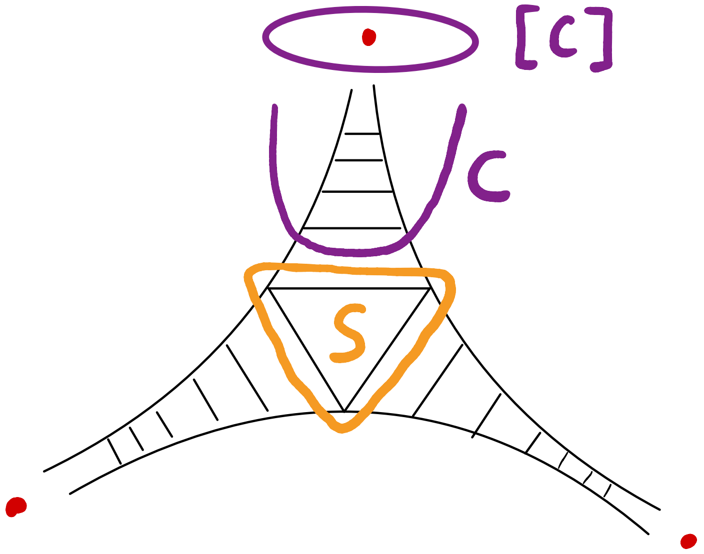

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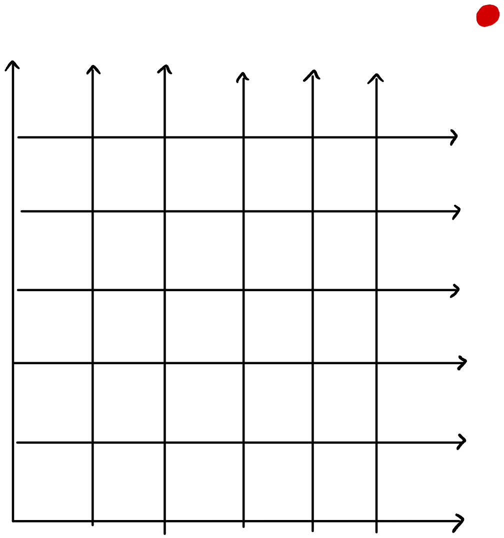

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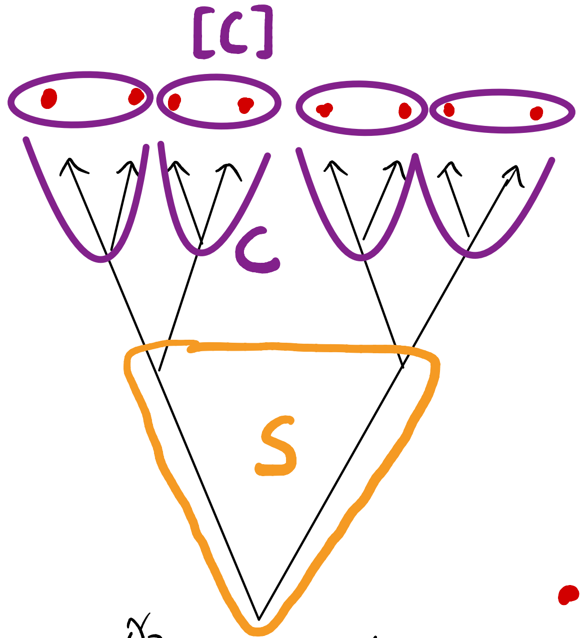

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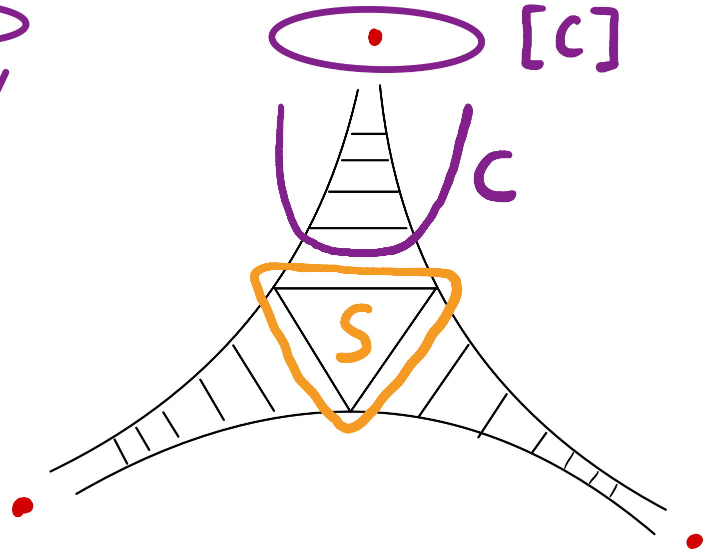

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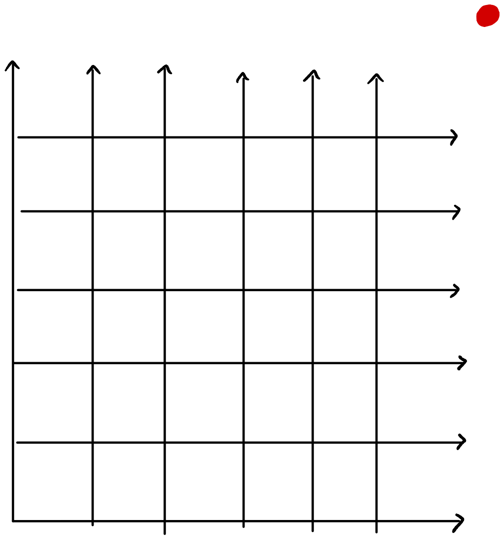

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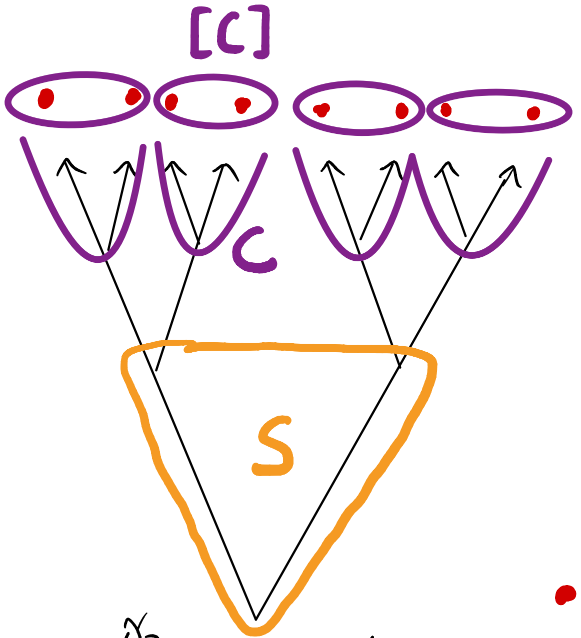

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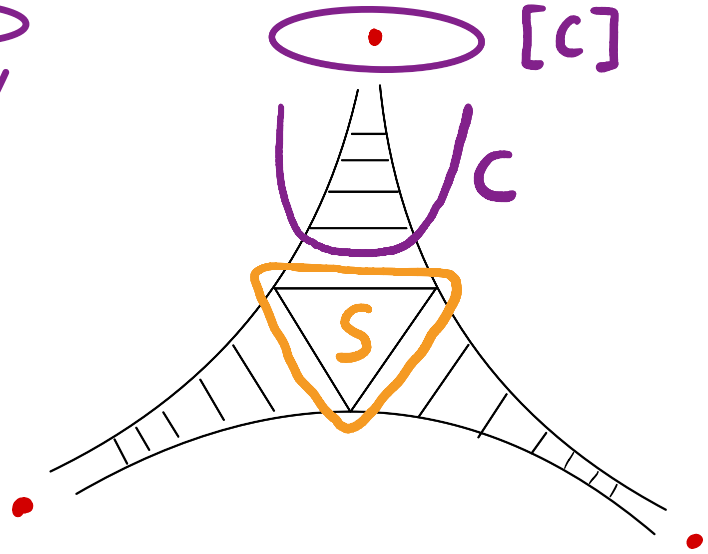

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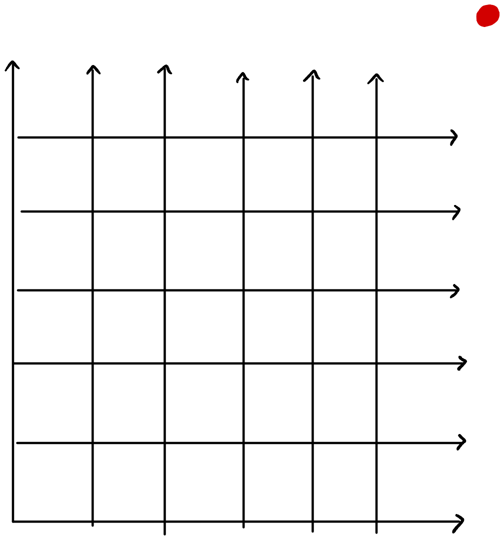

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↖ basis for a 0-dim top on $\Omega(G)$

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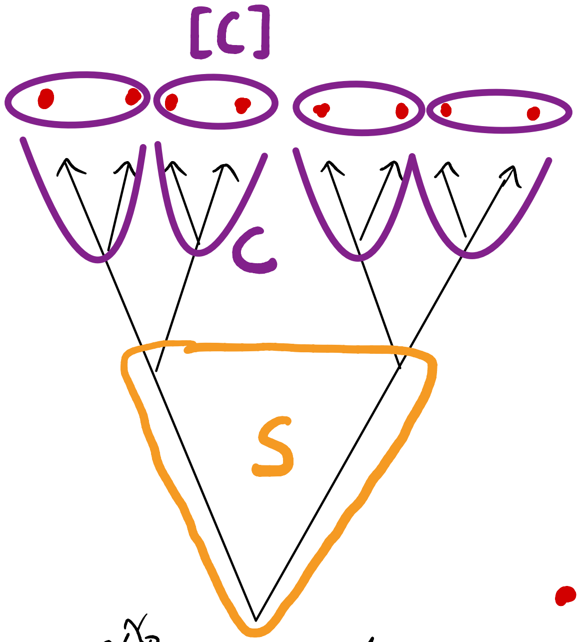

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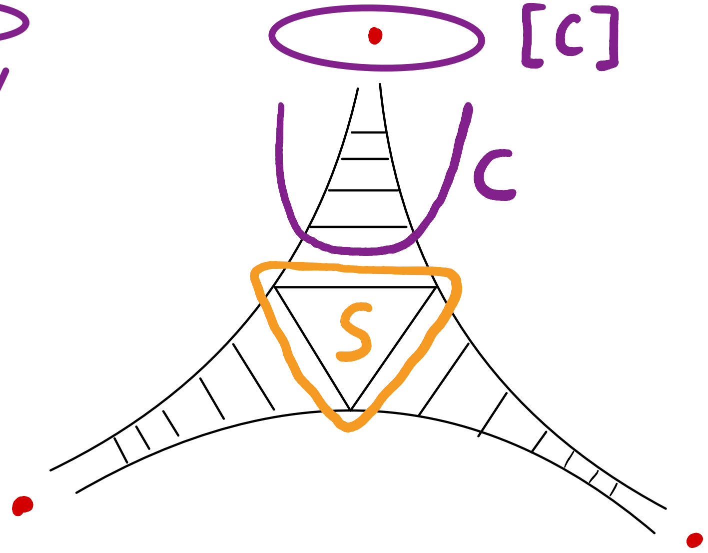

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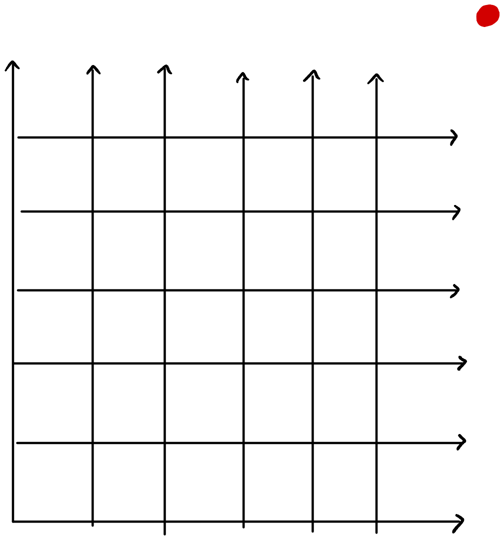

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TOPL. END SPACE PROBLEM (Diestel, 1992)

Which top^{l} spaces occur as end spaces of graphs?

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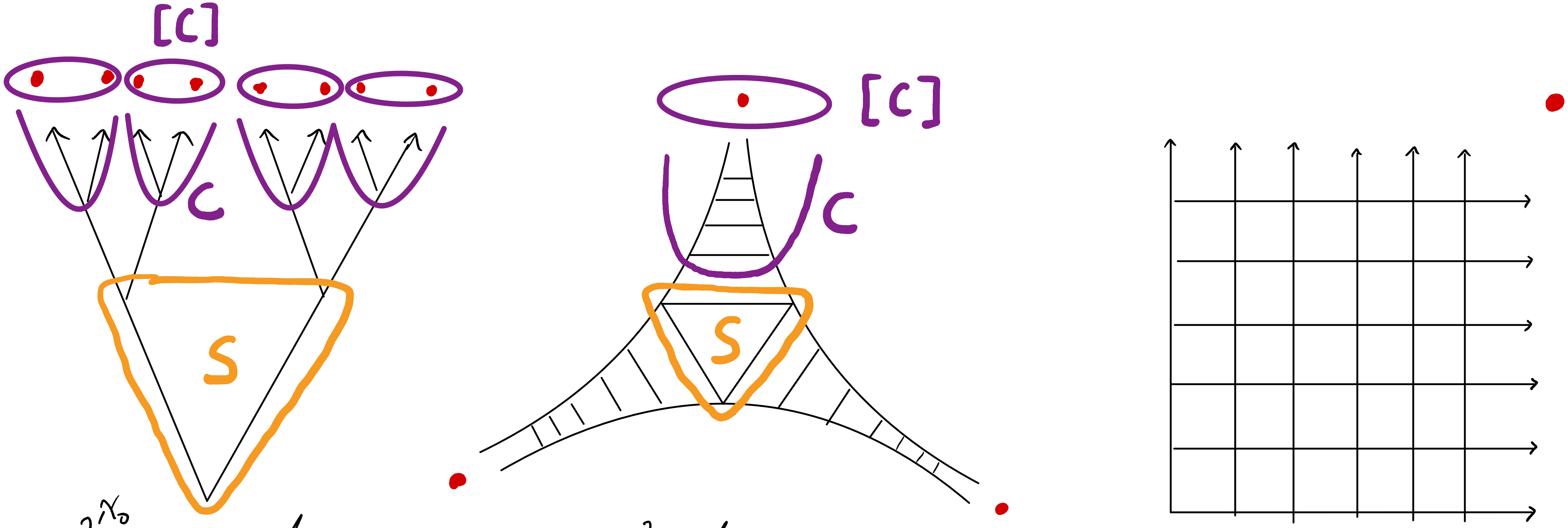

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- 3 ends
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TOPL. END SPACE PROBLEM (Diestel, 1992)

Which top^l spaces occur as end spaces of graphs?

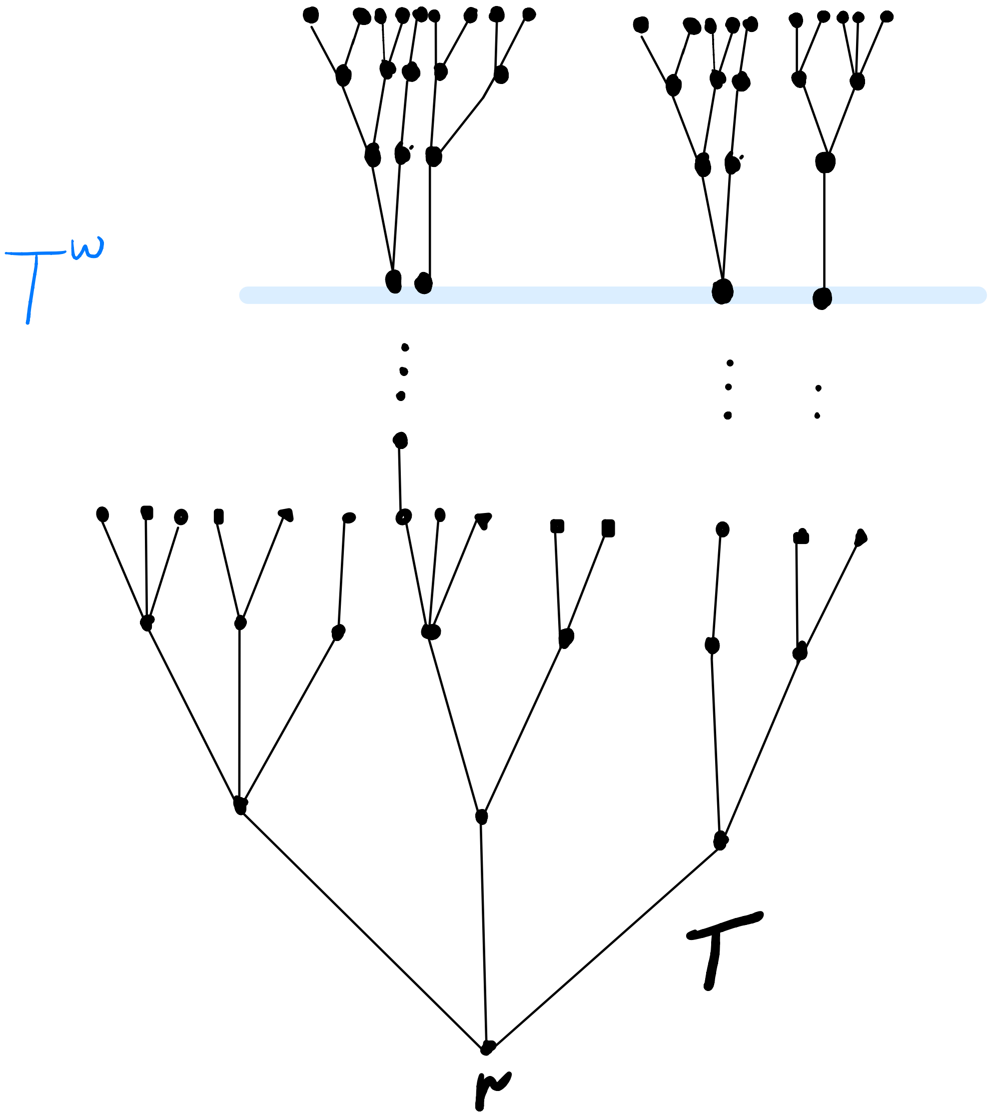
X end space of a tree ⟺ X completely ultrametr.

X end space of a graph ⟺ $\boxed{???}$

A REPRESENTATION THEOREM FOR END SPACES

(Kurkofka, Pitz 21^+)

BRANCH AND RAY SPACES OF ORDER TREES



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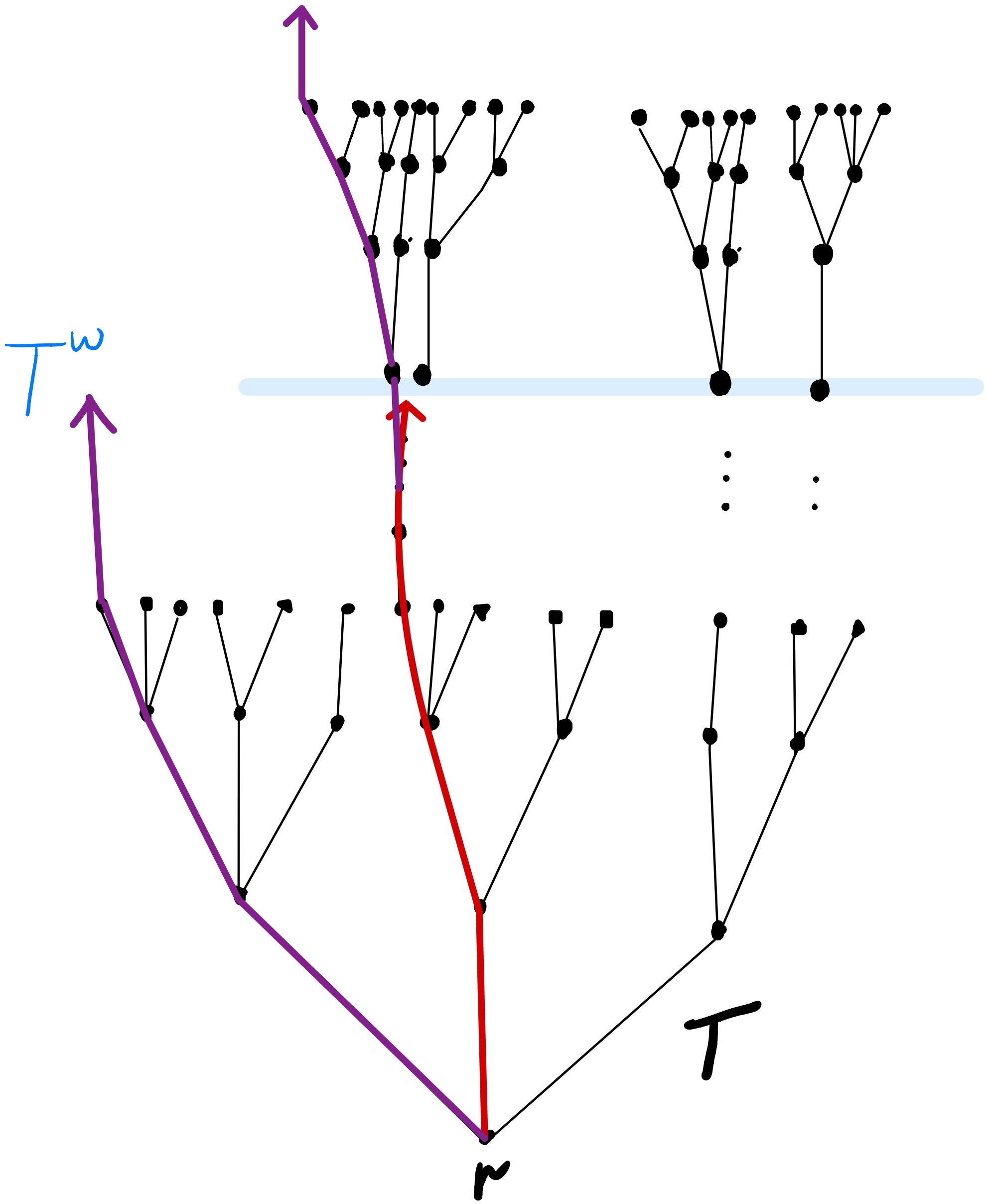

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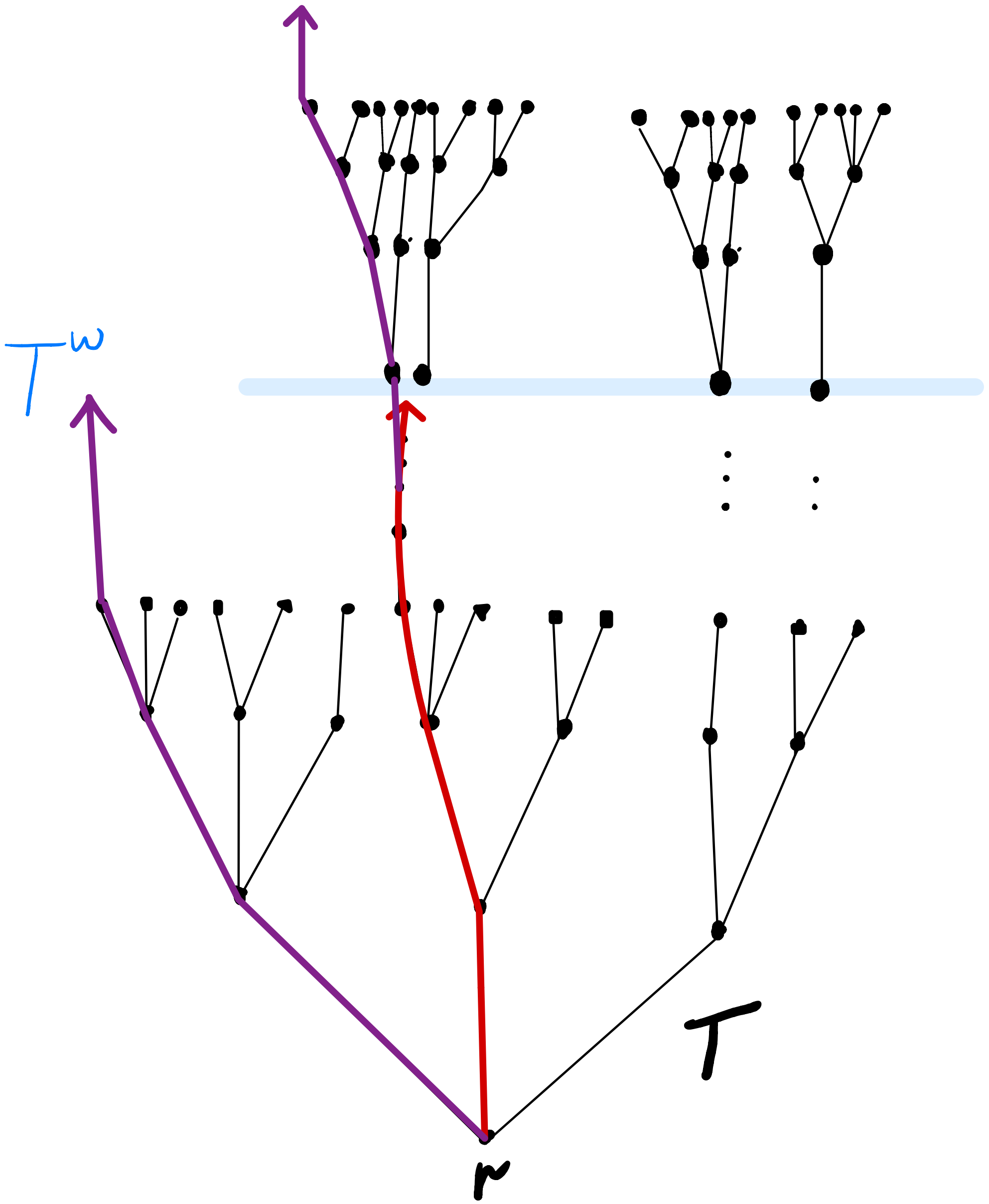

- T = rooted order tree, pruned, not nec. T_2
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- $\mathcal{R}(T) \geq \mathcal{B}(T)$

set of rays

set of branches

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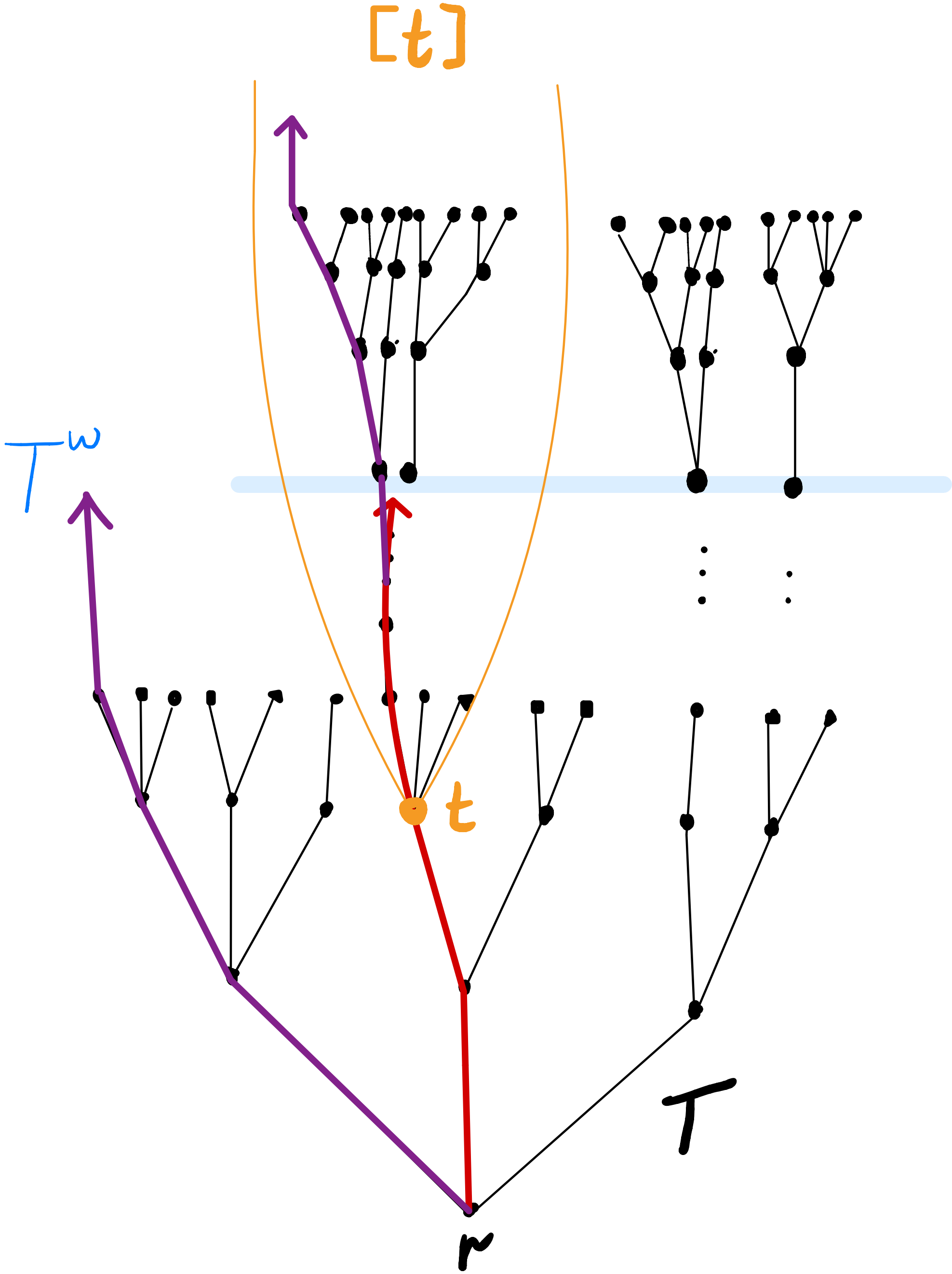

- T = rooted order tree, pruned, not nec. T_2
- Ray = down-closed chain without max. element
- Branch = $\subseteq$-max ray
- $\mathcal{R}(T) \supseteq \mathcal{B}(T)$
- clopen subbase

$$[t] = \{x \in \mathcal{R}(T) : t \in x\}$$

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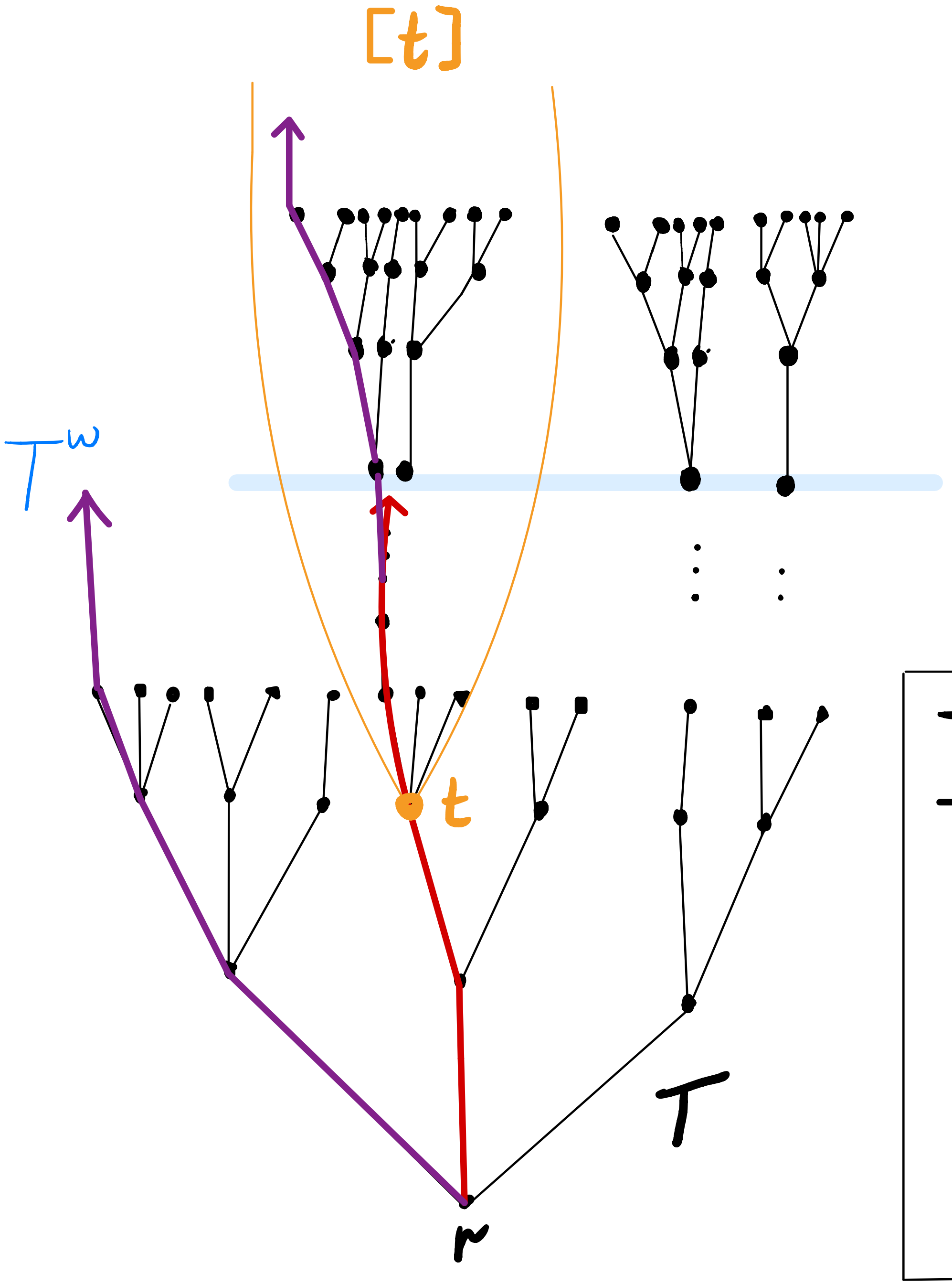

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THM (K+P '21): X top. space:

$X \cong \Omega(G)$ for graph G

$\Longleftrightarrow$

$X \cong \mathcal{R}(T)$ for special order tree T.

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⇐ „Builds on an idea of Diestel-Leader-Todorčević (2001)"

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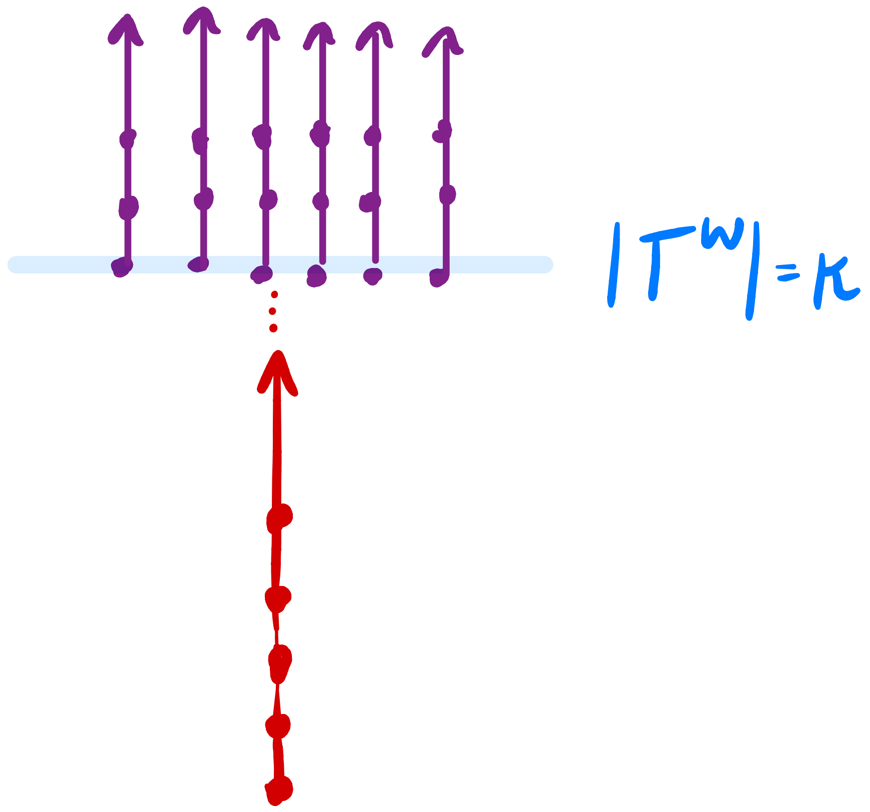

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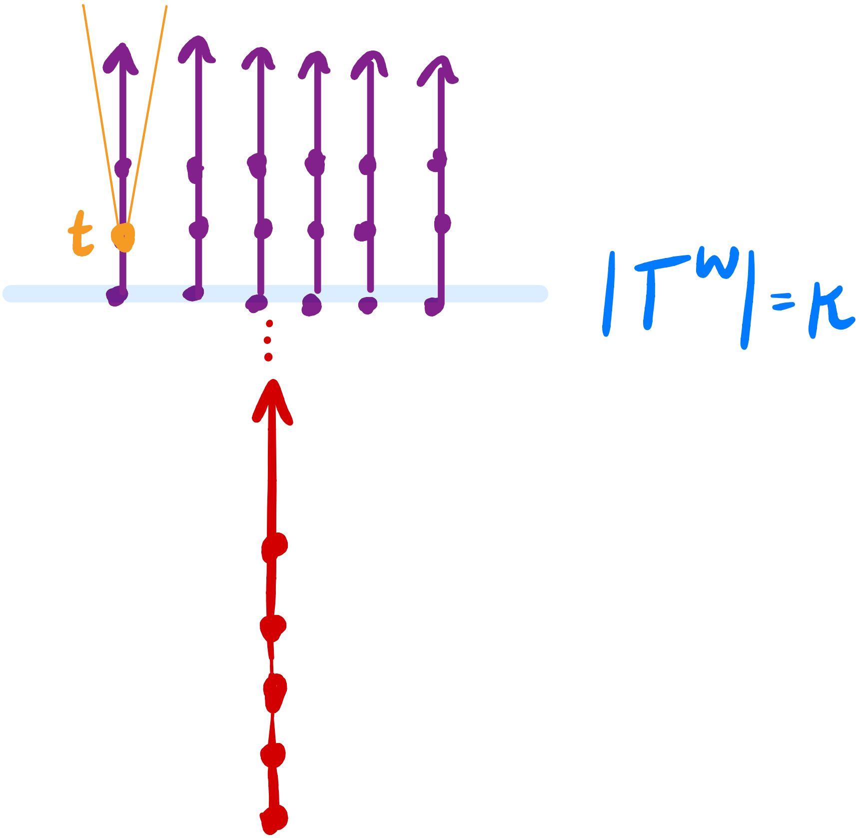

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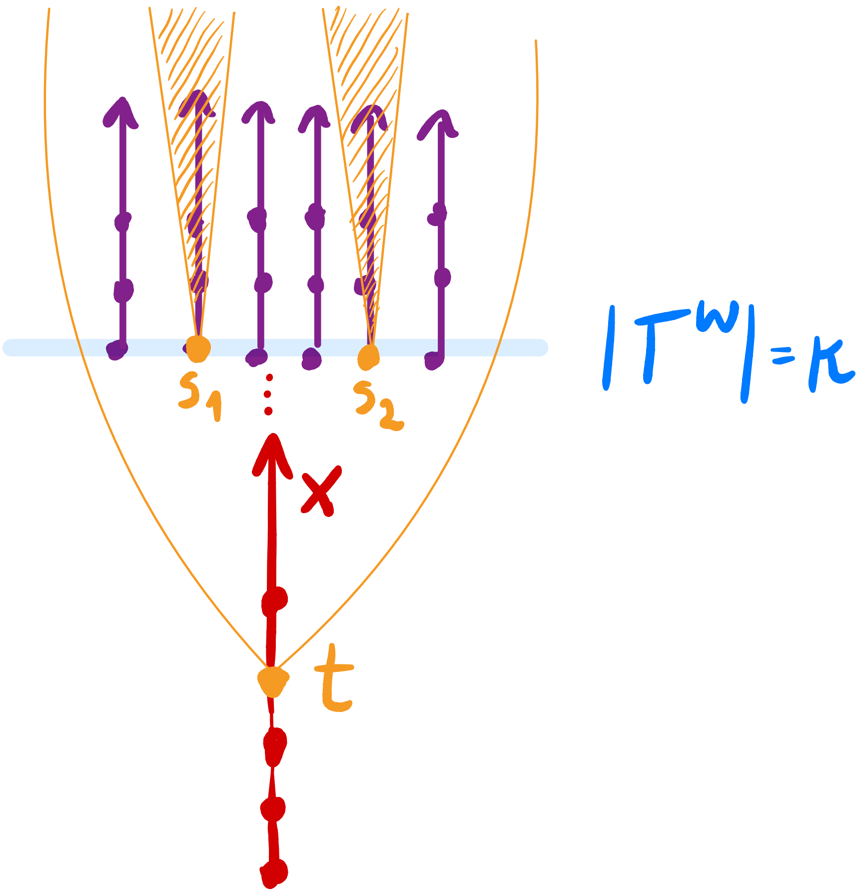

$B(T)$ = discrete of size κ

$R(T) = \alpha(B(T))$

↑ 1-pt compactification

Basis at x given by

$[t] \setminus ([s_1] \cup \dots \cup [s_n])$

where s_i are <u>tops</u> of ray x

PROPERTIES OF RAY SPACES

- for branch spaces: folklore
- for end spaces: Kurkofka, Melcher, Pitz '20

Every ray space is

- ultraparacpt
- monotonically normal
- base compact
- not necessarily Čech-complete

$\exists$ clopen base $\mathcal{B}$ s.t.
$\forall \hat{\mathcal{F}} \subseteq \mathcal{B}$ with f.i.p. : $\bigcap \hat{\mathcal{F}} \neq \emptyset$.

PROPERTIES OF RAY SPACES

Every ray space is ultraparacpt:

Let $\mathcal{U}$ be open cover of $R(T)$.

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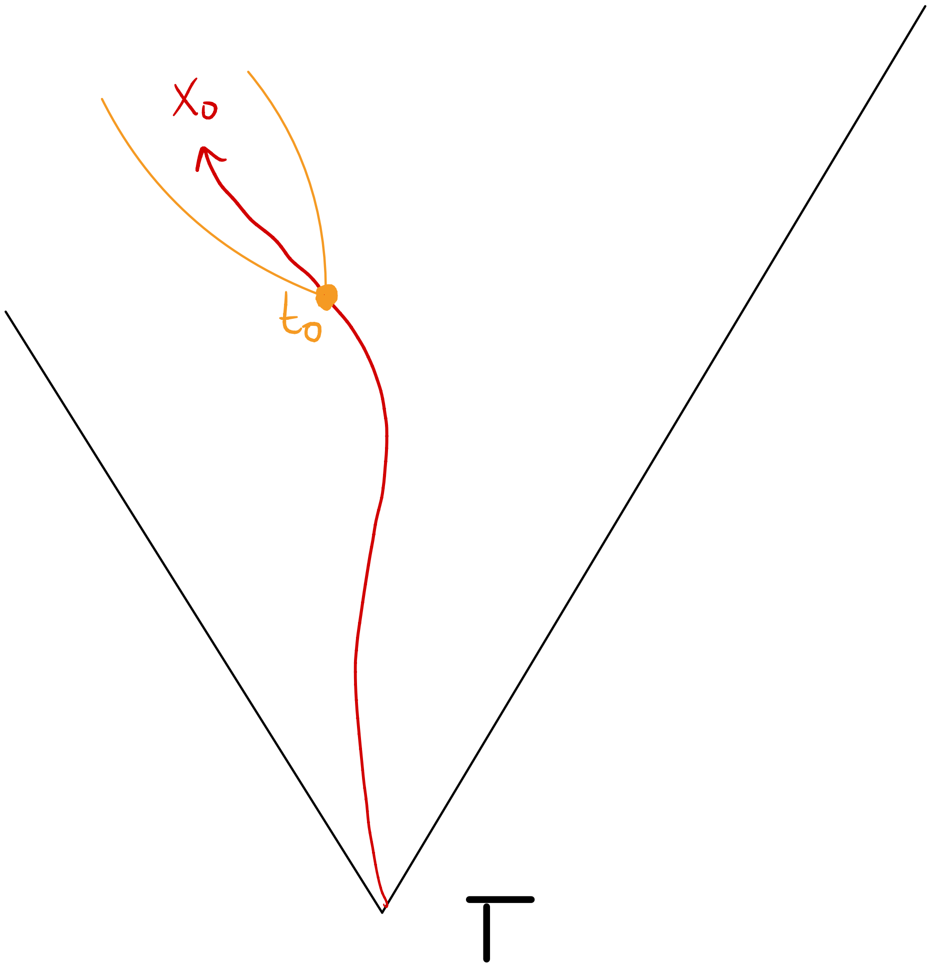

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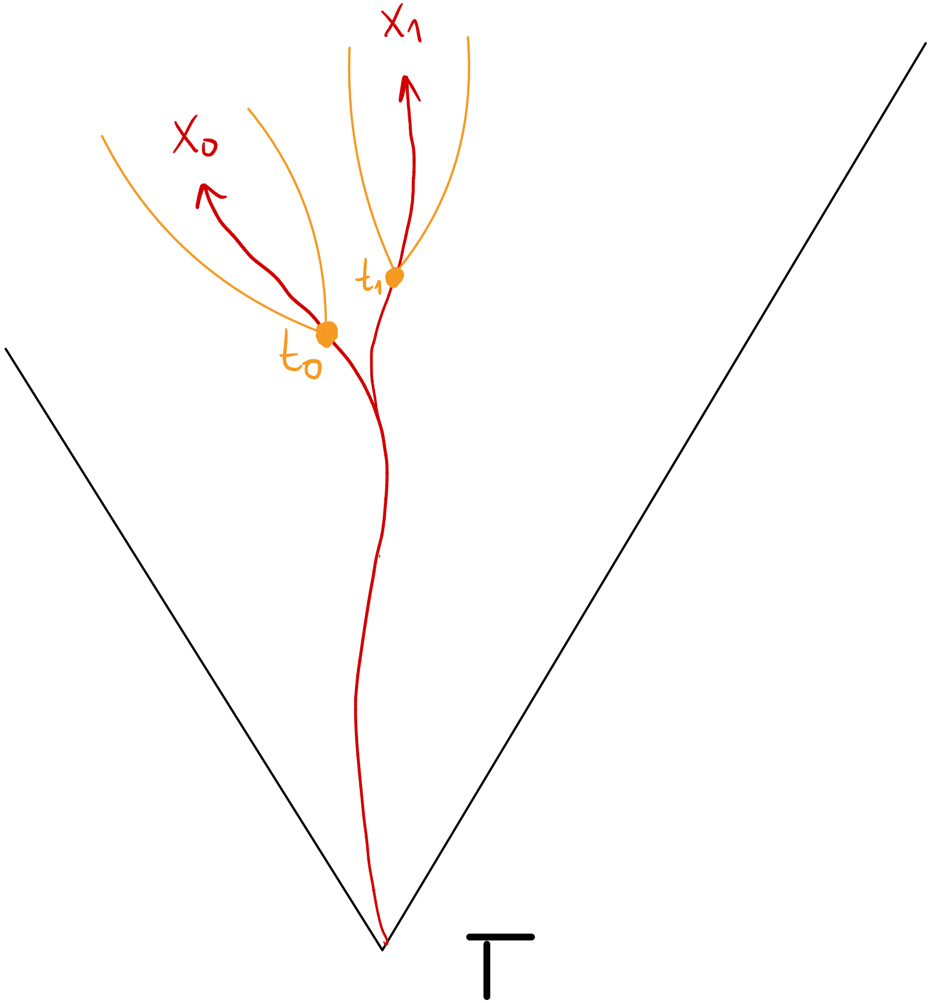

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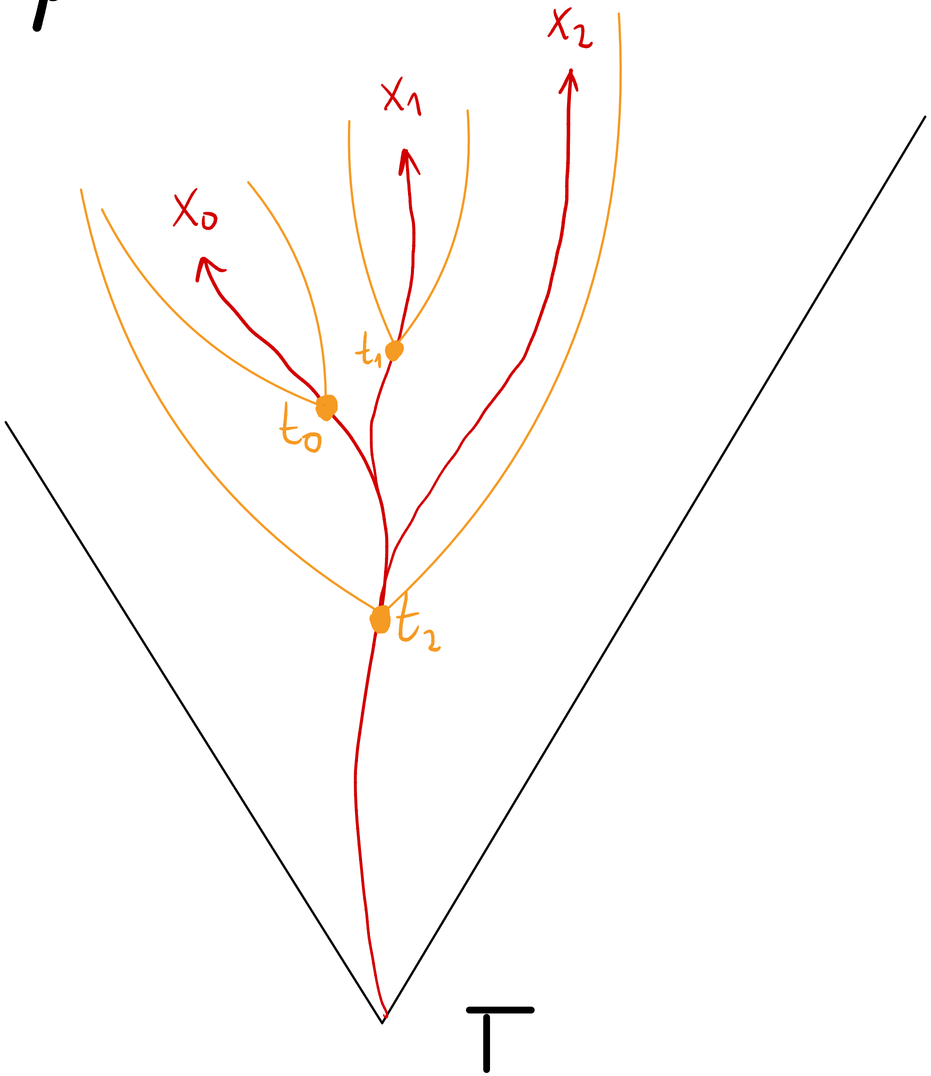

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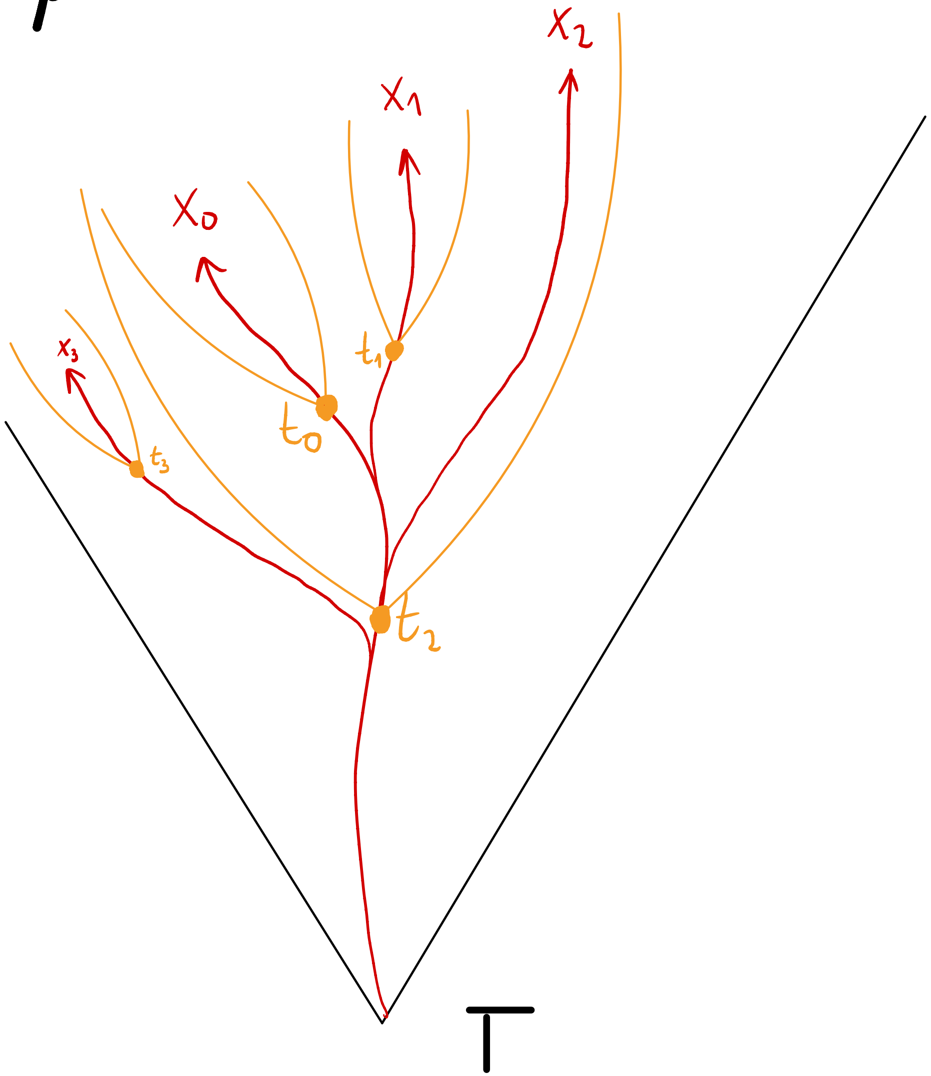

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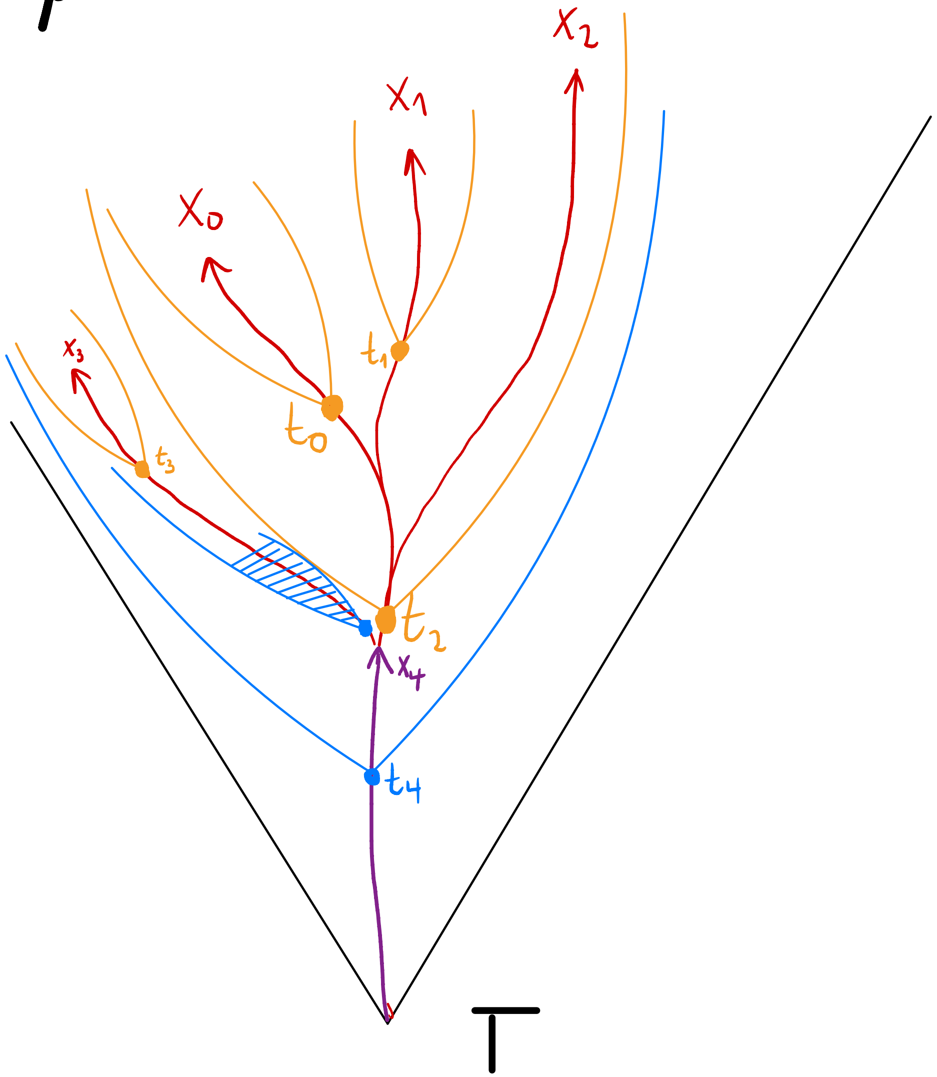
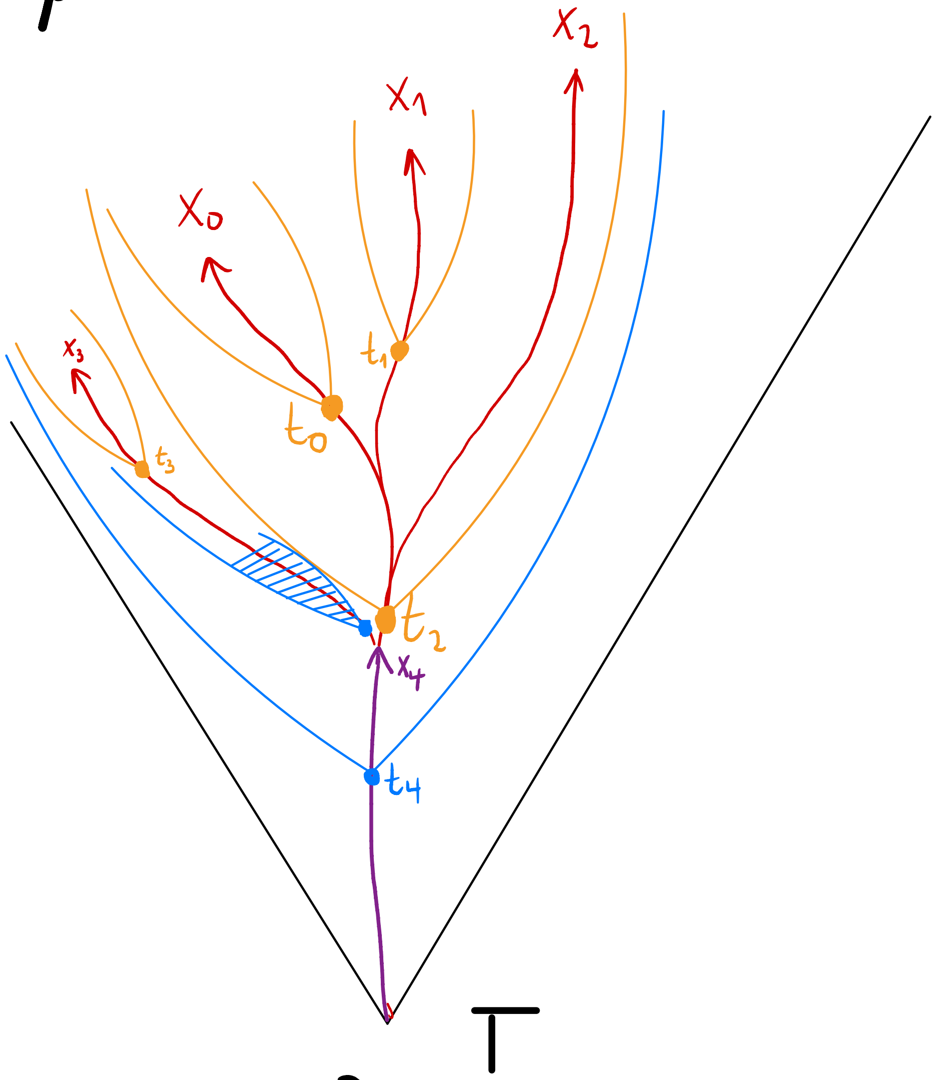


PROPERTIES OF RAY SPACES

Every ray space is ultraparacpt:

Let $\mathcal{U}$ be open cover of $R(T)$.

Want: open partition refinement $\mathcal{U}^* \preceq \mathcal{U}$.



Repeat as long as possible:

- pick an $\subseteq$-max uncovered ray $x_\alpha \in R(T)$
- pick basic open $x_\alpha \in V_\alpha$ $= [t_\alpha] \setminus ([s_1^\alpha] \cup \dots \cup [s_n^\alpha]) \subseteq V \in \mathcal{U}$

$\Rightarrow \mathcal{U}^* = \{ \subseteq\text{-max elmts in } \{V_\alpha : \alpha \in On\}\}$ works. $\square$

PROPERTIES OF RAY SPACES

Every ray space is

- ultraparacpt
- monotonically normal
- base compact

Every ray space of a special tree is

- hereditarily ultraparacpt
- contains a dense, completely metr. subset

↑ for branch spaces: Todorčerić, '81

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Every ray space of a special Aronszajn tree is

- Lindelöf
- ⇒ not metrizable

} for branch spaces:

- Todorčević '86,
- Funk, Lutzer '05.

A TOPL CHARACTERISATION OF RAY SPACES?

X homeomorphic to for T a	$R(T)$	$B(T)$
height-ω tree	X completely ultrametrizable	

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$\exists$ base $\mathcal{B}$ s.t. $\forall B_1, B_2 \in \mathcal{B}$:

$B_1 \cap B_2 \neq \emptyset \Rightarrow B_1 \subseteq B_2$ or $B_2 \subseteq B_1$.

A TOPL CHARACTERISATION OF RAY SPACES?

X homeomorphic to for T a	$R(T)$	$B(T)$
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Hausdorff order tree	?	

A TOPL CHARACTERISATION OF RAY SPACES?

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height-ω tree	X completely ultrametrizable	
order tree	?	T_2, base-cpt, nonarchimedean
Hausdorff order tree	?	(builds on Nyikos '99)
special order tree	???	+ $\exists\,\sigma$-disjoint base (builds on Funk/Lutzer '05)

For the history of the TOPL. END SPACE PROBLEM
& proof details of the REPRESENTATION THM

see <u>Kurkofka & Pitz: A REP. THEOREM FOR END SPACES</u>