Δ-spaces X and distinguished spaces $C_p(X)$

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My talk will be devoted to the main results obtained in the following joint works:

[1] J. C. Ferrando, J. Kąkol, A. Leiderman, S. A. Saxon, *Distinguished $C_p(X)$ spaces*, RACSAM **115:27** (2021).

[2] J. Kąkol, A. Leiderman, *A characterization of X for which spaces $C_p(X)$ are distinguished and its applications*, Proc. Amer. Math. Soc., series B, **8** (2021), 86–99.

[3] J. Kąkol, A. Leiderman, *Basic properties of X for which the space $C_p(X)$ is distinguished*, Proc. Amer. Math. Soc., series B, **8** (2021), 267–280.

[4] A. Leiderman, V.V. Tkachuk, *Pseudocompact Δ-spaces are often scattered*, Monatsh. Math. **197** (2022), 493–503.

[5] A. Leiderman, P. Szeptycki, *On Δ-spaces*, preprint.

1. Distinguished locally convex spaces

Following J. Dieudonné and L. Schwartz a locally convex space (lcs) E is called ***distinguished*** if every bounded subset of the bidual of E in the weak*-topology is contained in the closure of the weak*-topology of some bounded subset of E.

Equivalently, a lcs E is distinguished if and only if the strong dual of E (i.e. the topological dual of E endowed with the strong topology) is *barrelled*. A. Grothendieck (1954) proved that a metrizable lcs E is distinguished if and only if its strong dual is *bornological*.

All topological spaces X are assumed to be Tychonoff. We consider the locally convex space $C_p(X)$.

Theorem 2.1 ([1] L., Ferrando, Kąkol, Saxon)

For a Tychonoff space X, the following conditions are equivalent:

(1) $C_p(X)$ is distinguished.

(2) For every $f \in \mathbb{R}^X$ there is a bounded set $B \subset C_p(X)$ such that $f \in cl_{\mathbb{R}^X}(B)$.

(3) The strong dual of the space $C_p(X)$ carries the finest locally convex topology.

Denote by Δ the class of Tychonoff spaces X such that $C_p(X)$ is distinguished.

Major questions

(1) What is the internal description of $X \in \Delta$?

(2) What are the properties of $X \in \Delta$?

(3) To what extent the class Δ is invariant under the basic topological operations?

(a) A Q-set X is a subset of $\mathbb{R}$ such that each subset of X is F_σ, or, equivalently, each subset of X is G_δ in X.

(b) A Δ-set X is a subset of $\mathbb{R}$ such that for every decreasing sequence $\{D_n : n \in \omega\}$ of subsets of X with $\bigcap_{n\in\omega} D_n = \emptyset$, there is a decreasing sequence $\{V_n : n \in \omega\}$ consisting of open subsets of X such that $D_n \subset V_n$ for every n, and again with $\bigcap_{n\in\omega} V_n = \emptyset$.

Status of the existence of Q-sets

It is not difficult to show that each Q-set must be a Δ-set. The existence of an uncountable Q-set is one of the fundamental set-theoretical problems. F. Hausdorff (1933) showed that

(1) If X is an uncountable Q-set, then $|X| < \mathfrak{c} = 2^{\aleph_0}$. So, assuming the Continuum Hypothesis (CH) there are no uncountable Q-sets.

On the other hand,

(2) Martin's Axiom plus the negation of the Continuum Hypothesis (MA $+\neg$CH) implies that every subset $X \subset \mathbb{R}$ of cardinality less than $\mathfrak{c}$ is a Q-set (Martin-Solovay (1970), M.E. Rudin (1977)).

Status of the existence of Δ-sets

The definition of a Δ-set of reals was given by G.M. Reed and then improved by E. van Douwen in 1977.

(3) No Δ-set X can have cardinality $\mathfrak{c}$ (T. Przymusiński (1977)). Hence, under MA, every subset of $\mathbb{R}$ that is a Δ-set is also a Q-set.

(4) Question: is it consistent that there exists a Δ-set $X \subset \mathbb{R}$ that is not a Q-set?
R. Knight published a paper (1993) aiming to show that the answer is "Yes". However, up to now no expert can confirm that the proof is correct.

Main new notion

A Tychonoff space X is called a Δ-***space*** if for every decreasing sequence $\{D_n : n \in \omega\}$ of subsets of X with $\bigcap_{n\in\omega} D_n = \emptyset$, there is a decreasing sequence $\{V_n : n \in \omega\}$ consisting of open subsets of X such that $D_n \subset V_n$ for every n, and again with $\bigcap_{n\in\omega} V_n = \emptyset$.

A collection of sets $\{U_\gamma : \gamma \in \Gamma\}$ is called an ***expansion*** of a collection of sets $\{X_\gamma : \gamma \in \Gamma\}$ in X if $X_\gamma \subseteq U_\gamma \subseteq X$ for every index $\gamma \in \Gamma$.

Theorem 3.1 ([2] L., Kąkol)

For a Tychonoff space X, the following conditions are equivalent:

(1) $C_p(X)$ is distinguished.

(2) Any countable disjoint collection of subsets of X admits a point-finite open expansion in X.

(3) X is a Δ-space.

Independently and simultaneously an analogous description of distinguished C_p-spaces (but formulated in different terms) appeared in the paper of J.C. Ferrando and S. Saxon.

4. First examples of Δ-spaces and examples of non-Δ-spaces

Definition 4.1

A family $\{\mathcal{N}_x : x \in X\}$ of subsets of a Tychonoff space X is called a ***point-finite neighborhood assignment*** for X if each $\mathcal{N}_x$ is an open neighborhood of x and for each $u \in X$ the set $X_u = \{x \in X : u \in \mathcal{N}_x\}$ is finite.

Theorem 4.2 ([1])

If X admits a point-finite neighborhood assignment $\{\mathcal{N}_x : x \in X\}$, then $C_p(X)$ is distinguished.

Corollary 4.3 ([1])

If X is countable, then X admits a point-finite neighborhood assignment, hence $C_p(X)$ is distinguished.

Corollary 4.4 ([1])

If X has only finitely many non-isolated points, then X admits a point-finite neighborhood assignment, hence $C_p(X)$ is distinguished.

Theorem 4.5 ([1])

A compact space X admits a point-finite neighborhood assignment iff X is a **scattered Eberlein compact space**.
So, if X is a scattered Eberlein compact space, then $C_p(X)$ is distinguished.

Class Δ

Recall that the class of all Δ-spaces by Δ, i.e. $X \in \Delta$ iff $C_p(X)$ is distinguished.

Proposition 4.6 ([5] L., Szeptycki)

(a) Let X be a hereditarily separable space. If $|X| = \mathfrak{c}$, then $X \notin \Delta$.

(b) Let X be a separable hereditarily Lindelöf space. If $|X| = \mathfrak{c}$, then $X \notin \Delta$.

Proposition 4.7 ([5])

Let X be any subspace of the Sorgenfrey line S with $|X| = \mathfrak{c}$. Then $X \notin \Delta$.

Separable metrizable Δ-spaces

(1) The existence of an uncountable separable metrizable space X such that $C_p(X)$ is distinguished, is independent of ZFC.

(2) There exists an uncountable separable metrizable space X such that $C_p(X)$ is distinguished, if and only if there exists a separable countably paracompact nonnormal Moore space.

5. Compact (and close to compact) Δ-spaces

Theorem 5.1 ([2])

Every countably compact Δ-space is scattered.

Theorem 5.2 ([2])

Every Čech-complete Δ-space is scattered.

Theorem 5.2 extends a well-known result of B. Knaster and K. Urbanik stating that every countable Čech-complete space is scattered.

However, we showed that there do exist scattered compact spaces which are not in the class Δ.

Theorem 5.3 ([2])

$[0, \omega_1)$ is not in the class Δ. Hence, the compact scattered space $[0, \omega_1]$ is not a Δ-space.

Theorem 5.4 ([3] L., Kąkol)

Every compact Δ-space has countable tightness.

A very natural question arises whether Theorem 5.8 can be generalized for countably compact spaces. A positive answer follows from the Proper Forcing Axiom (PFA), due to the celebrated results of Z. Balogh.

Theorem 5.5 ([3])

Assume (PFA). Then

(1) Every countably compact Δ-space is compact.

(2) Every countably compact Δ-space has countable tightness.

(3) Every countably compact Δ-space (hence, every compact Δ-space) is sequential.

Problem 1

Does Theorem 5.5 hold in ZFC alone?

The known proof of Theorem 5.1 fails if we assume only that X is a pseudocompact (and non-normal) space.

Problem 2

Let X be a pseudocompact Δ-space. Is it true that X is scattered?

Towards the solution of Problem 1 we have the following results.

Theorem 5.6 ([4])

Every pseudocompact Δ-space of countable tightness is scattered.

However,

Theorem 5.7 ([5])

There exists a pseudocompact Δ-space (and scattered) but with uncountable tightness.

If D is the set of natural numbers $\mathbb{N}$ and $\mathcal{A}$ is a maximal almost disjoint (MAD) family of subsets of $D = \mathbb{N}$, then the corresponding **Isbell–Mrówka space** is denoted by $\Psi(\mathcal{A})$.

Theorem 5.10 ([2])

Let X be the one-point compactification of an Isbell–Mrówka space $\Psi(\mathcal{A})$. Then X is a compact Δ-space, while X is not an Eberlein compact space.

MAD families on ω_1

Let D be ω_1 and $\mathcal{A}$ be a MAD family of countable subsets of ω_1. It is unknown whether the space $\Psi(D, \mathcal{A})$ in this case can be a Δ-space for some MAD family $\mathcal{A}$ in some model of ZFC.

Theorem 5.11 ([5])

In ZFC there exists a MAD family of countable subsets of $D = \omega_1$ such that $\Psi(D, \mathcal{A})$ is not a Δ-space. Denote by X the one-point compactification of the above locally compact space $\Psi(D, \mathcal{A})$. Then X is a scattered compact space with the Cantor-Bendixson rank equal to 2, and $X \notin \Delta$.

Problem 3

Does there exist in ZFC a MAD family of countable subsets of D with $|D| \geq \aleph_1$ such that $\Psi(D, \mathcal{A})$ is a Δ-space?

6. Invariance of the class Δ under basic operations

In this section we consider the question whether the class Δ is invariant under the following basic topological operations: subspaces, continuous images, quotient continuous images, countable unions, finite products.

Proposition 6.1

Let Y be any subspace of X. If X belongs to the class Δ, then Y also belongs to the class Δ.

The last result can be reversed, assuming that $X \setminus Y$ is finite.

Proposition 6.2 ([2])

Suppose that Y is a Δ-subspace of a space X such that $X \setminus Y$ is finite. Then X is a Δ-space.

Proposition 6.3 ([1])

The Δ-property can be destroyed by adding a countable set to a discrete space.

Proposition 6.4 ([4])

Suppose that Y is a Δ-subspace of a space X such that $X \setminus Y$ is countable and scattered. Then X is a Δ-space.

Proposition 6.5 ([2])

There exists in ZFC a MAD family $\mathcal{A}$ on $\mathbb{N}$ such that the corresponding Isbell–Mrówka space $\Psi(\mathcal{A})$ admits a continuous mapping onto the closed interval $[0,1]$. Thus, the class Δ is not invariant under continuous images even for first-countable separable locally compact pseudocompact spaces.

Theorem 6.6 ([3])

Let X be a Δ-space and $\varphi : X \to Y$ be a continuous surjection such that $\varphi(F)$ is an F_σ-set in Y for every closed set $F \subset X$. Then Y is also a Δ-space.

Theorem 6.7 ([3])

Assume that X is a countable union of closed subsets X_n, where each X_n belongs to the class Δ. Then X also belongs to Δ. In particular, a countable union of compact Δ-spaces is also a Δ-space.

Proposition 6.8 ([2])

Any σ-scattered metrizable space belongs to the class Δ.

The following question concerning compact Δ-spaces is motivated by the observation that all known examples can be seen to be countable unions of scattered Eberlein compacta.

Problem 4

Are all compact Δ-spaces countable unions of scattered Eberlein compacta?

Problem 5

Let Z be the product of two compact Δ-spaces X and Y. Is Z a Δ-space?

We have a partial positive result for products of Δ-spaces.

Proposition 6.10 ([3])

Let Z be the product of a Δ-space X with a strongly σ-discrete space (in particular, with a countable space) Y. Then Z also is a Δ-space.

7. Relation of ℓ-dominance and the class Δ

Following A. V. Arkhangel'skii, we say that a space Y is ℓ-***dominated*** by a space X if $C_p(X)$ can be mapped linearly and continuously onto $C_p(Y)$. There are many topological properties which are invariant under defined above relation, and there are many which are not.

The main goal of this section is to study the following question: *Which topological properties related to being a Δ-space are preserved by the relation of ℓ-dominance*?

Theorem 7.1 ([3])

Assume that Y is ℓ-dominated by X. If X is a Δ-space, then Y also is a Δ-space.

Using a similar idea we prove

Theorem 7.2 ([3])

Let X and Y be metrizable spaces, and assume that Y is ℓ-dominated by X. If X is a Q-space, then Y also is a Q-space.

Proposition 7.4 ([3])

Assume that Y is ℓ-dominated by X.

(1) If X is an Eberlein compact, then Y also is an Eberlein compact.

(2) If X is a scattered Eberlein compact, then Y also is a scattered Eberlein compact.

For Banach spaces $C(X)$ the following problem is unsolved.

Problem 6

Let X and Y be two compact spaces and assume that the Banach spaces $C(X)$ and $C(Y)$ are isomorphic. Is it true that Y is a Δ-space provided X is a Δ-space?

Problem 7

Let X be the one-point compactification of an Isbell–Mrówka space $\Psi(\mathcal{A})$. Assume that Y is a compact space such that the Banach spaces $C(X)$ and $C(Y)$ are isomorphic. Is it true that Y is a Δ-space?

Thank you!