Big Ramsey degrees and infinite languages

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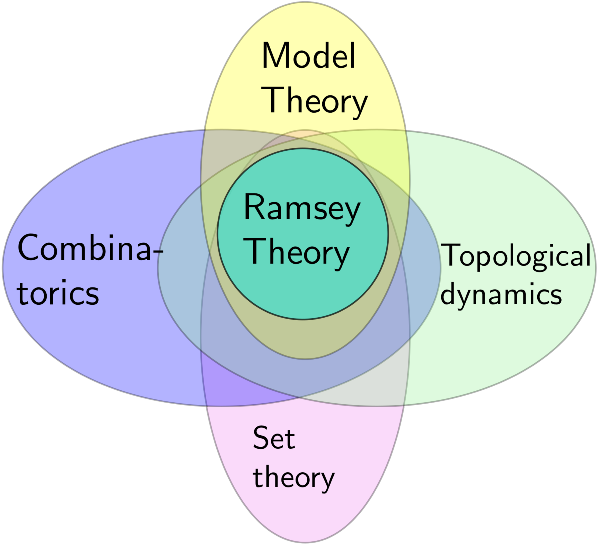
TOPOSYM 2022

Joint work with Samuel Braunfeld, David Chodounský, Noé de Rancourt, Jan Hubička and Jamal Kawach
$$$ GAČR 21-10775S, ERC 810115 $$$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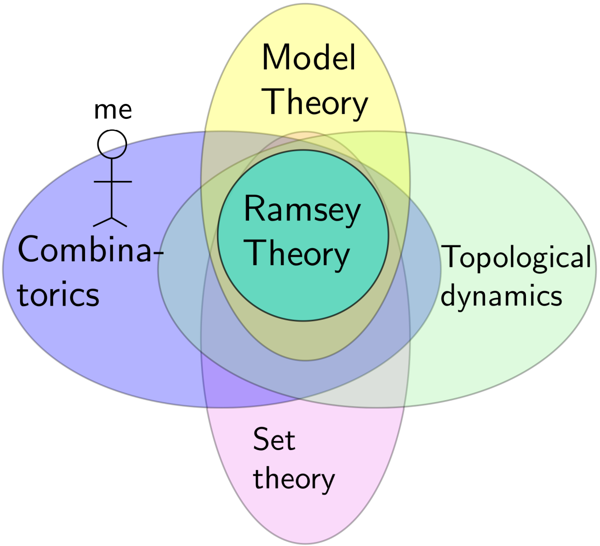
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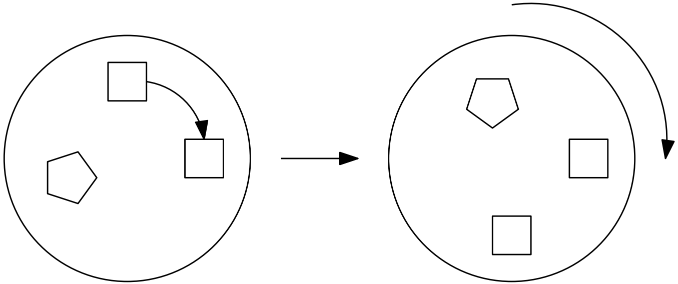
Homogeneous structures

A structure **A** is (ultra)homogeneous if every isomorphism of its finite substructures can be extended to an automorphism of **A**.



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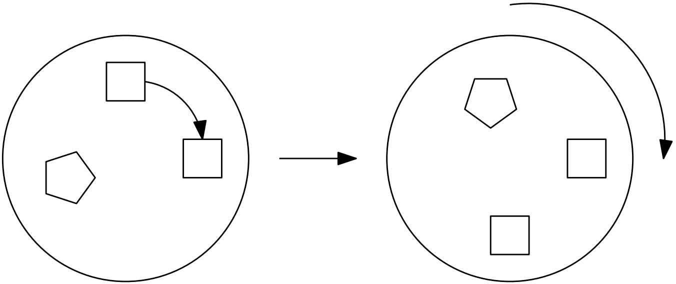

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- The countable random (Rado) graph
- $\mathbb{F}_2^\infty$

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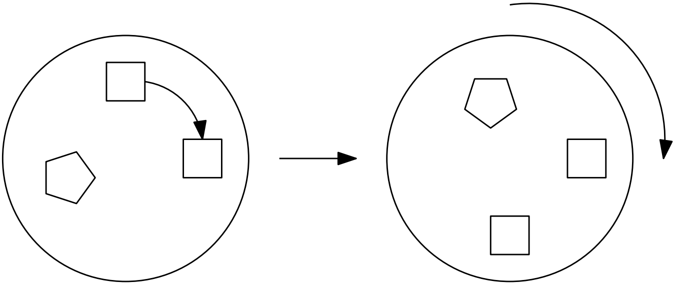

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The age of **A** is the class of all finite substructures of **A**.

Topological dynamics

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Theorem (Kechris, Pestov, Todorcevic'05)

Let $\mathbf{A}$ *be homogeneous. Then* $\mathrm{Aut}(\mathbf{A})$ *is extremely amenable if and only if* $\mathrm{Age}(\mathbf{A})$ *has the Ramsey property.*

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Class $\mathcal{C}$ of finite structures has the Ramsey property if for every $\mathbf{A}, \mathbf{B} \in \mathcal{C}$ there is $\mathbf{C} \in \mathcal{C}$ such that $\mathbf{C} \longrightarrow (\mathbf{B})^{\mathbf{A}}_{2,1}$.

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One can use Ramsey property to compute universal minimal flows [KPT, Nguyen Van The, Zucker].

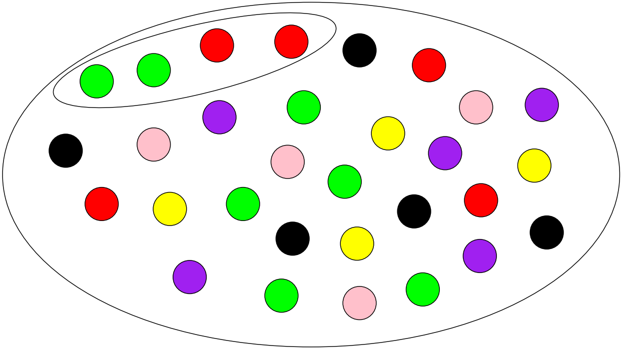
Partition arrow

Let $\mathbf{A}, \mathbf{B}, \mathbf{C}$ be structures and let $k, t \in \omega$. We denote by

$$\mathbf{C} \longrightarrow (\mathbf{B})^{\mathbf{A}}_{k,t}$$

the statement that for every colouring $c\colon \mathrm{Emb}(\mathbf{A}, \mathbf{C}) \to k$ there is $f \in \mathrm{Emb}(\mathbf{B}, \mathbf{C})$ such that

$$|c(f \circ \mathrm{Emb}(\mathbf{A}, \mathbf{C}))| \leq t.$$



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Theorem (Ramsey's theorem)

$$(\forall p \in \omega)(\forall k \in \omega)\ \omega \longrightarrow (\omega)^{p}_{k,1}$$

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Given $\mathbf{M}$ and $\mathbf{A} \in \mathrm{Age}(\mathbf{M})$, the big Ramsey degree of $\mathbf{A}$ in $\mathbf{M}$ is the least $t \in \omega \cup \{\omega\}$ such that for every $k \in \omega$ it holds that $\mathbf{M} \longrightarrow (\mathbf{M})^{\mathbf{A}}_{k,t}$.

Understanding big Ramsey degrees is helpful for computing universal completion flows [Zucker'19].

Big Ramsey degrees

Question

Given a (countably infinite) structure $\mathbf{M}$ and a finite colouring of its substructures isomorphic to $\mathbf{A}$, is there a monochromatic (oligochromatic) copy of $\mathbf{M}$?

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- Sets (Ramsey's theorem) more precisely $(\omega, \leq)$
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- $(\mathbb{Q}, \leq) \not\longrightarrow (\mathbb{Q})^{\cdot\cdot}_{2,1}$

Sierpiński's colouring

Fix an enumeration $\trianglelefteq$ of $\mathbb{Q}$. Given $a < b \in \mathbb{Q}$, put $\chi(a,b) = 0$ if $a \triangleleft b$ and $\chi(a,b) = 1$ otherwise.

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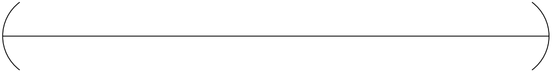

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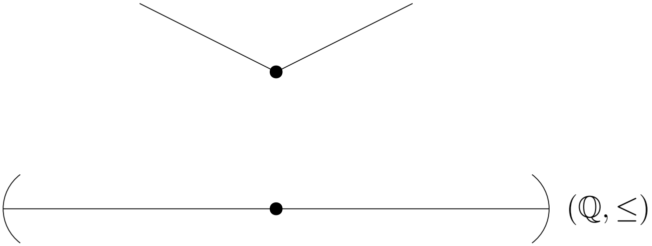

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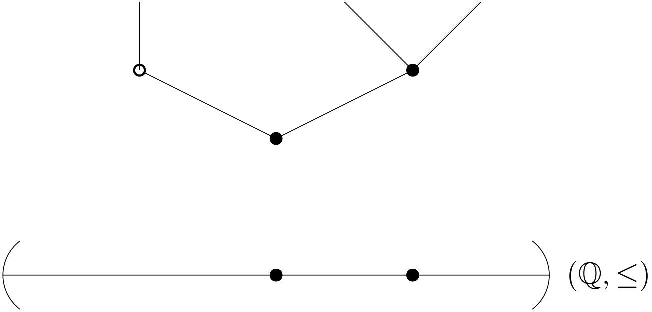

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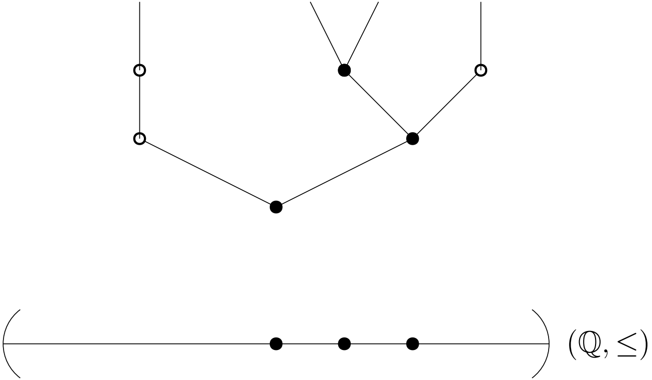

Colouring of n-element subsets

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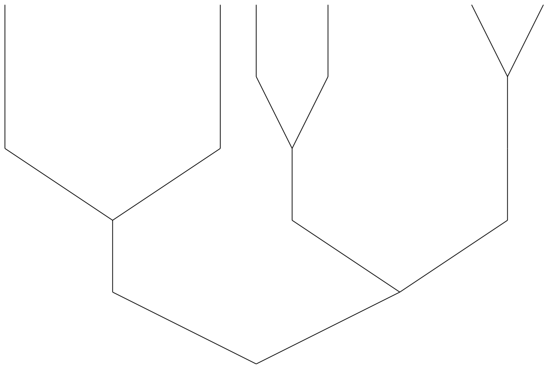

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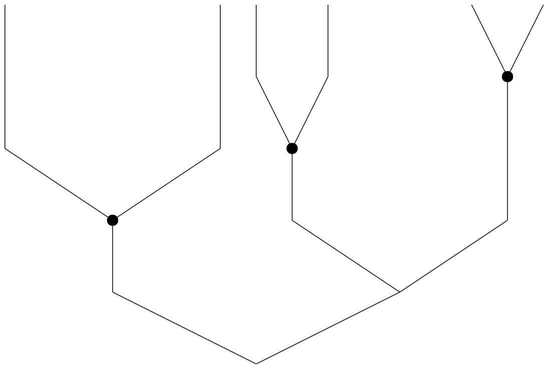

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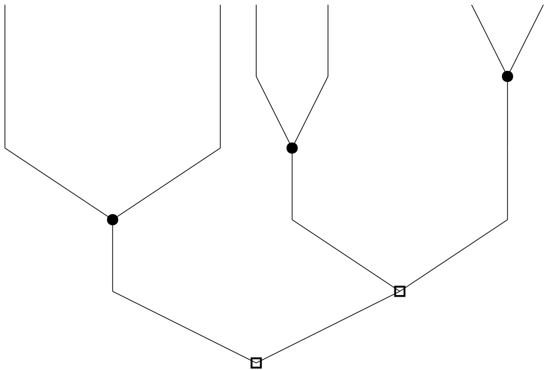

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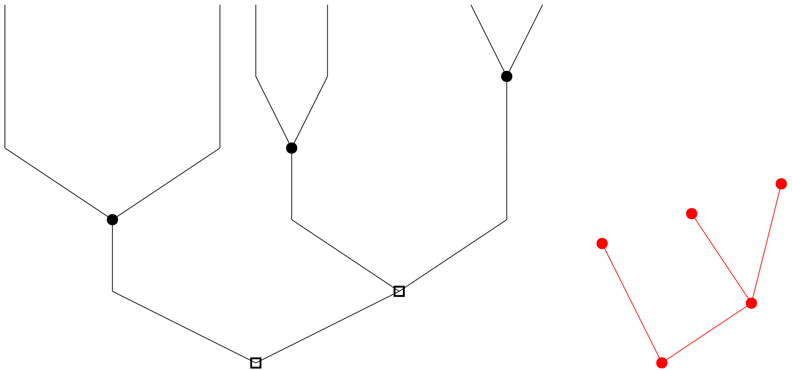

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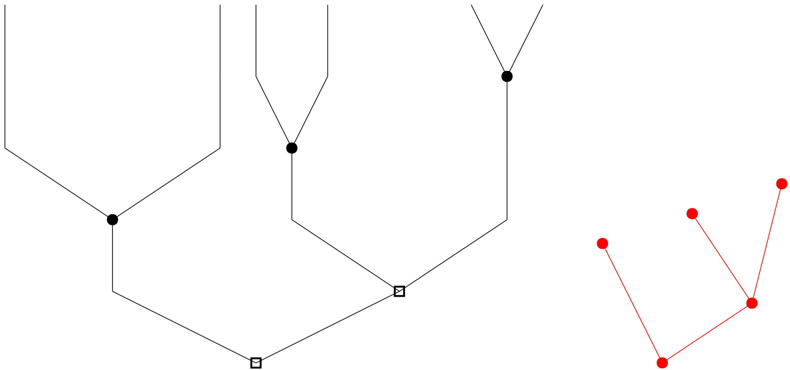

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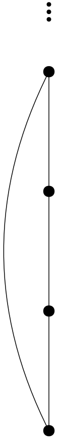
[Devlin 1979] found a sub-colouring of χ_n which is bad and universal at the same time, giving the big Ramsey degrees for $\mathbb{Q}$. (The big Ramsey degree of the n-element order is $\tan^{(2n-1)}(0)$.)

The random graph



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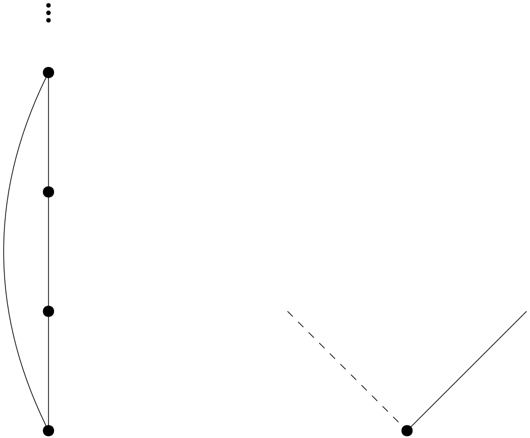

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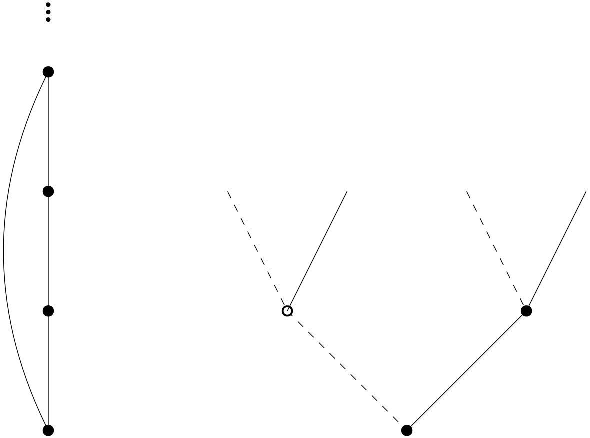

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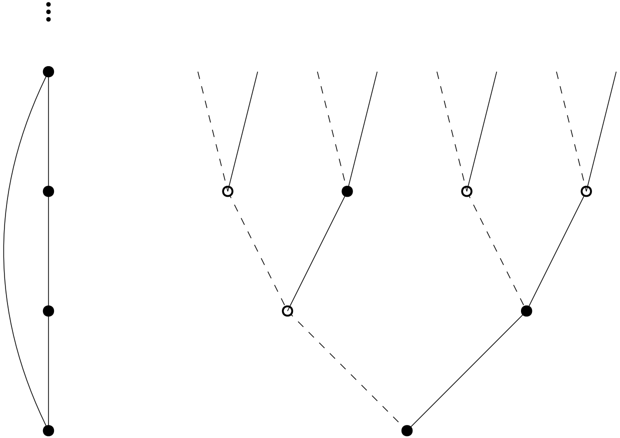

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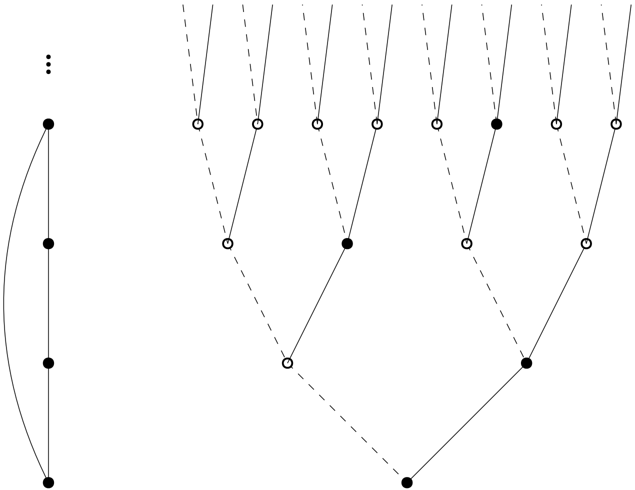

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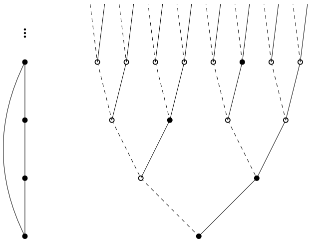

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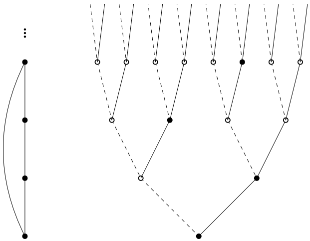

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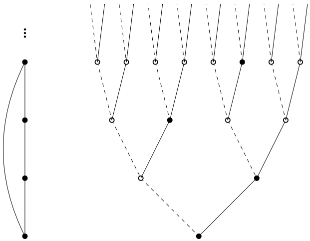

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χ_n is a universal colouring. (Uses Milliken's tree theorem.)

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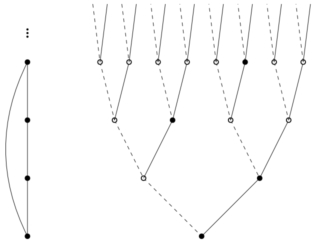

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Theorem (Laflamme, Sauer, Vuksanović 2006)

There is a sub-colouring of χ_n which is both universal and bad.

Examples

- $(\mathbb{N}, \leq)$ [Ramsey, 1928] (Ramsey)
- $(\mathbb{Q}, \leq)$ [Laver 1969, Devlin 1979, Galvin] (Milliken)
- random graph [Sauer 2006, Laflamme, Sauer, Vuksanović 2006] (Milliken)
- K_n-free graphs [Dobrinen *2016*, Balko, Chodounský, Dobrinen, Hubička, K, Vena, Zucker 2021] (Custom, using forcing)
- Generic partial order [Hubička 2020] (Carlson–Simpson)
- Metric spaces with finitely many distances [Balko, Chodounský, Hubička, K, Nešetřil, Vena 2020] (Carlson–Simpson)
- Free amalgamation classes in a finite binary language [Zucker 2020, Balko, Chodounský, Dobrinen, Hubička, K, Vena, Zucker 2021] (Custom, using forcing)
- 3-uniform hypergraphs [Balko, Chodounský, Hubička, K, Vena 2020] (Product Milliken)
- ...

The 3-uniform hypergraph

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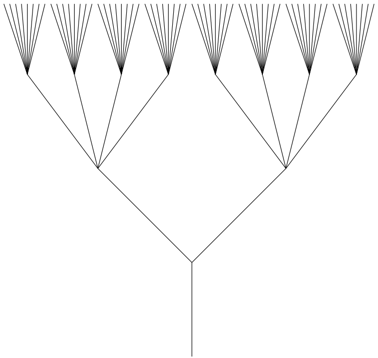
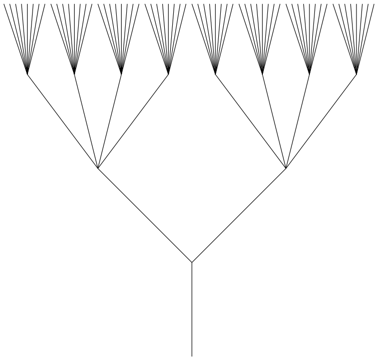


Tree of (1)-types

The 3-uniform hypergraph



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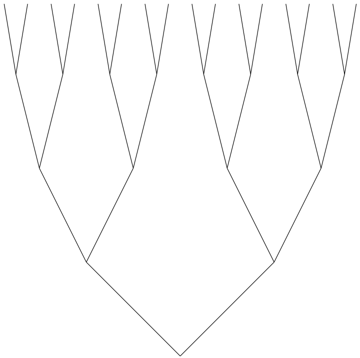
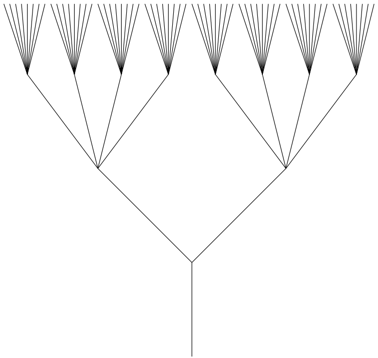
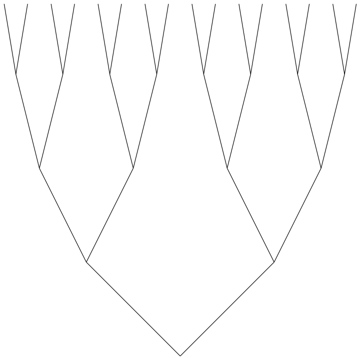


“Tree of (2)-types”

The 3-uniform hypergraph





Theorem (Balko, Chodounský, Hubička, K, Vena 2020)

The colourings by shapes in the product of the trees are universal.

(Uses the product Milliken tree theorem.)

Theorem (BChdRHKK, 2022)

Let L be a countable relational language with finitely many unaries *and let $\mathbb{H}$ be the Fraïssé limit of the class of all finite L-structures where each relation is injective. TFAE*

1. $\mathbb{H}$ *has finite big Ramsey degrees (there are universal colourings with finite domain),*
2. $\mathbb{H}$ *is ω-categorical,*
3. *L has finitely many relations of each arity,*
4. *the tree of (1)-types is finitely branching,*
5. *the tree of (n)-types is finitely branching for every n.*

(Using the product Milliken tree theorem.)

Lower bound

Proposition

Let T be the tree of all finite sequences of natural numbers. There is a colouring $c\colon T \to \omega$ such that whenever T' is a strong subtree of T of infinite height then $c[T'] = \omega$.

Proof.

1. Given $t \in T$, put $w(t) = |t| + \sum_{i<|t|} t(i)$ and define $\ell(t)$ to be the least ℓ such that $w(t \restriction_{\ell}) \geq |t|$.
2. Put $c(t) = w(t \restriction_{\ell(t)}) - |t|$.
3. Let T' be a strong subtree of T of infinite height, let r be the root of t' and let $n \in \omega$ be such that $n > w(r)$ and $T' \cap T(n) \neq \emptyset$. Put $k = n - w(r) - 1$, fix a colour $x \in \omega$ and find $t \in T'$ such that $|t| = n$ and $r^\frown(k + x) \sqsubseteq t$.
4. Now, $w(r) < n$, so $\ell(t) > |r|$. On the other hand, $w(r^\frown(k + x)) = k + x + w(r) + 1 = n + x \geq n$, hence $\ell(t) = |r| + 1$ and $c(t) = x$.

□

Thank you!