# Embeddings of the pseudo-arc into some Spaces

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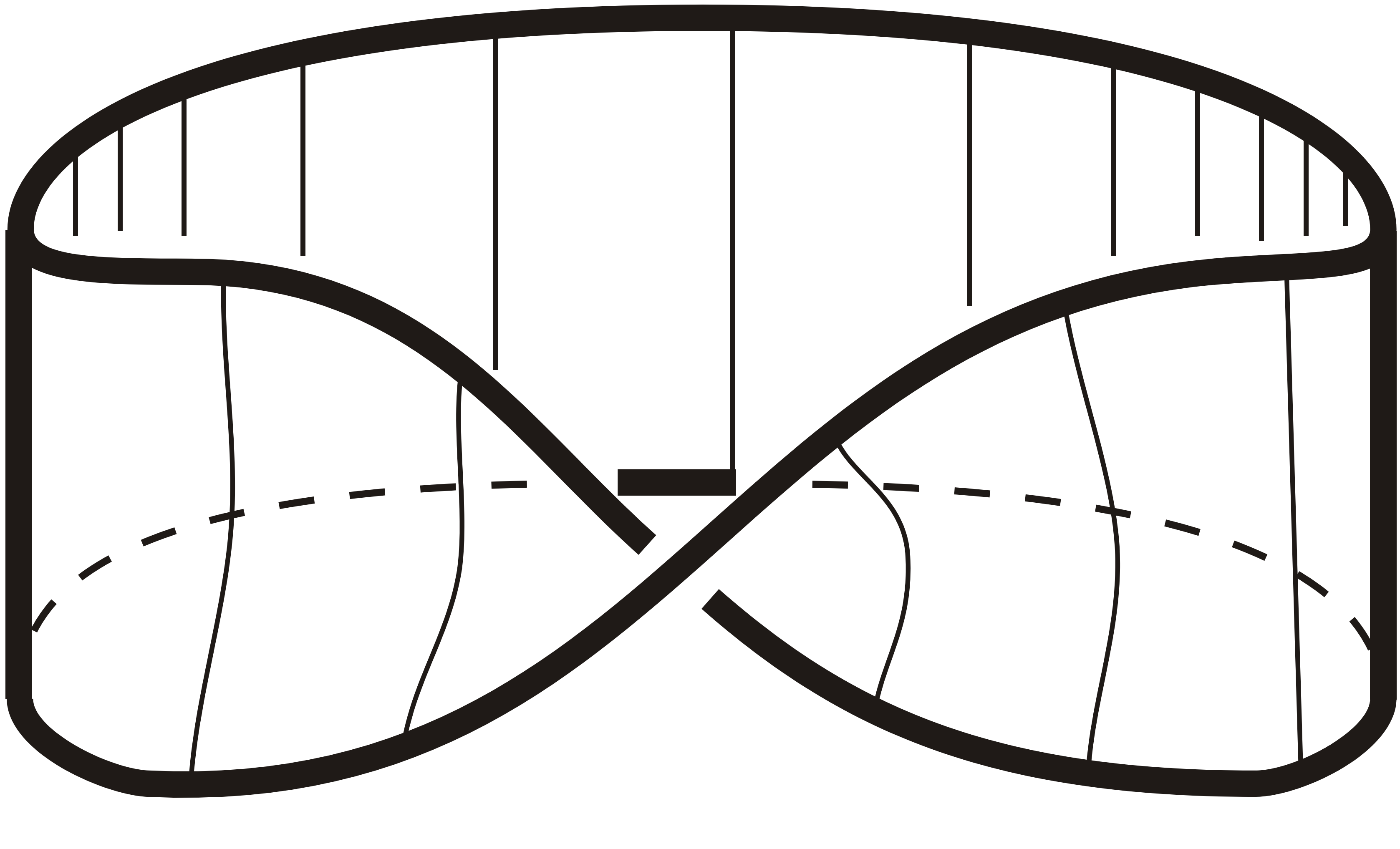
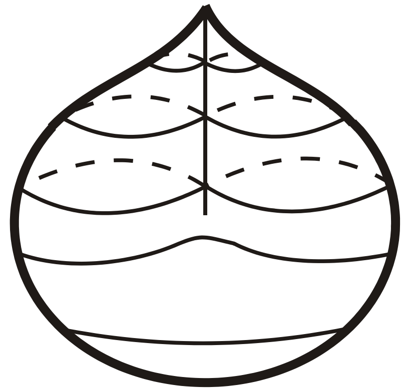
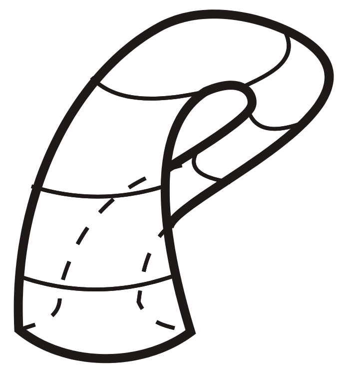
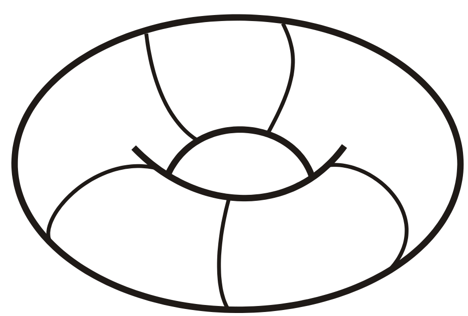
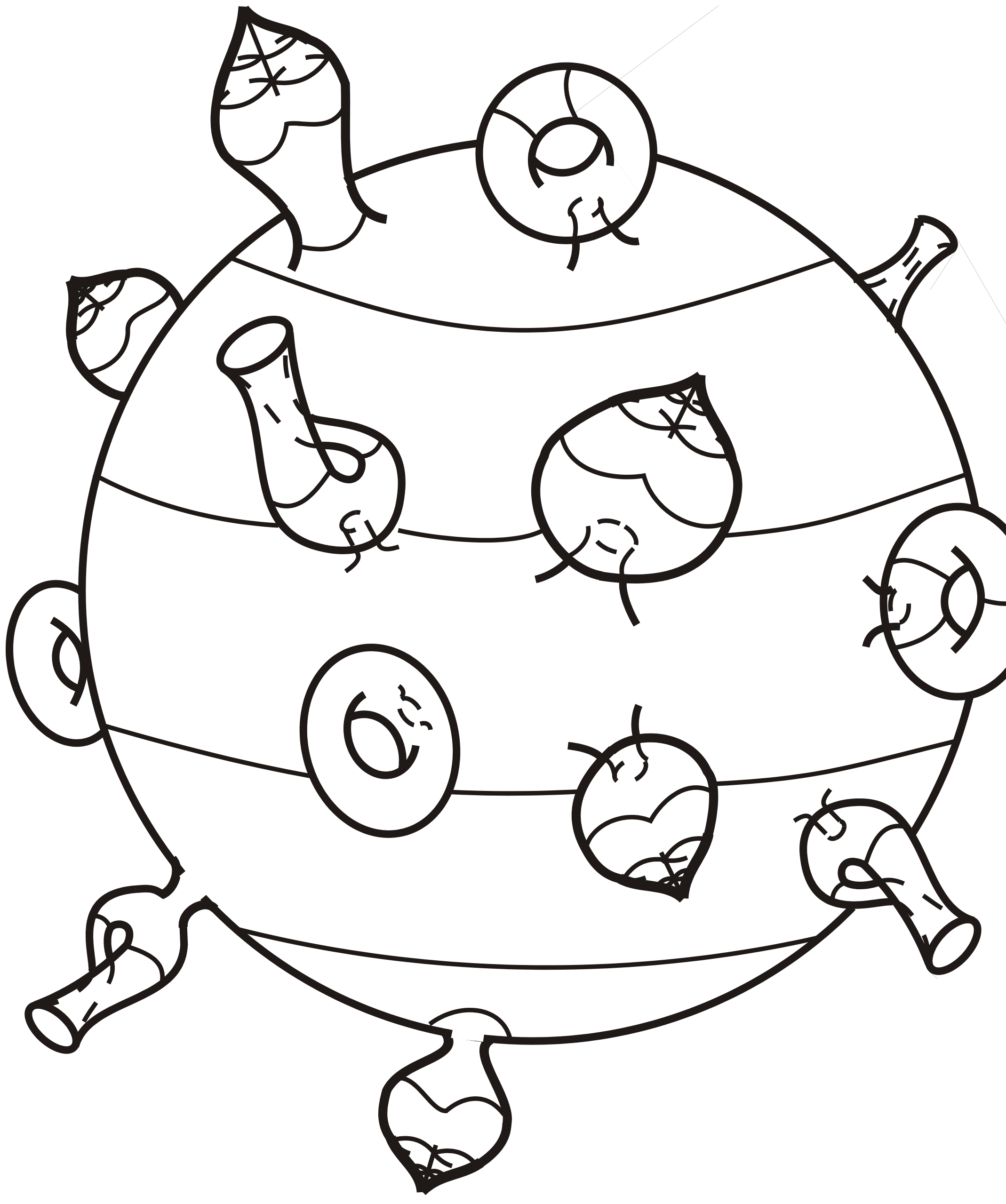
A ***continuum*** is a compact connected metric space with more than one point

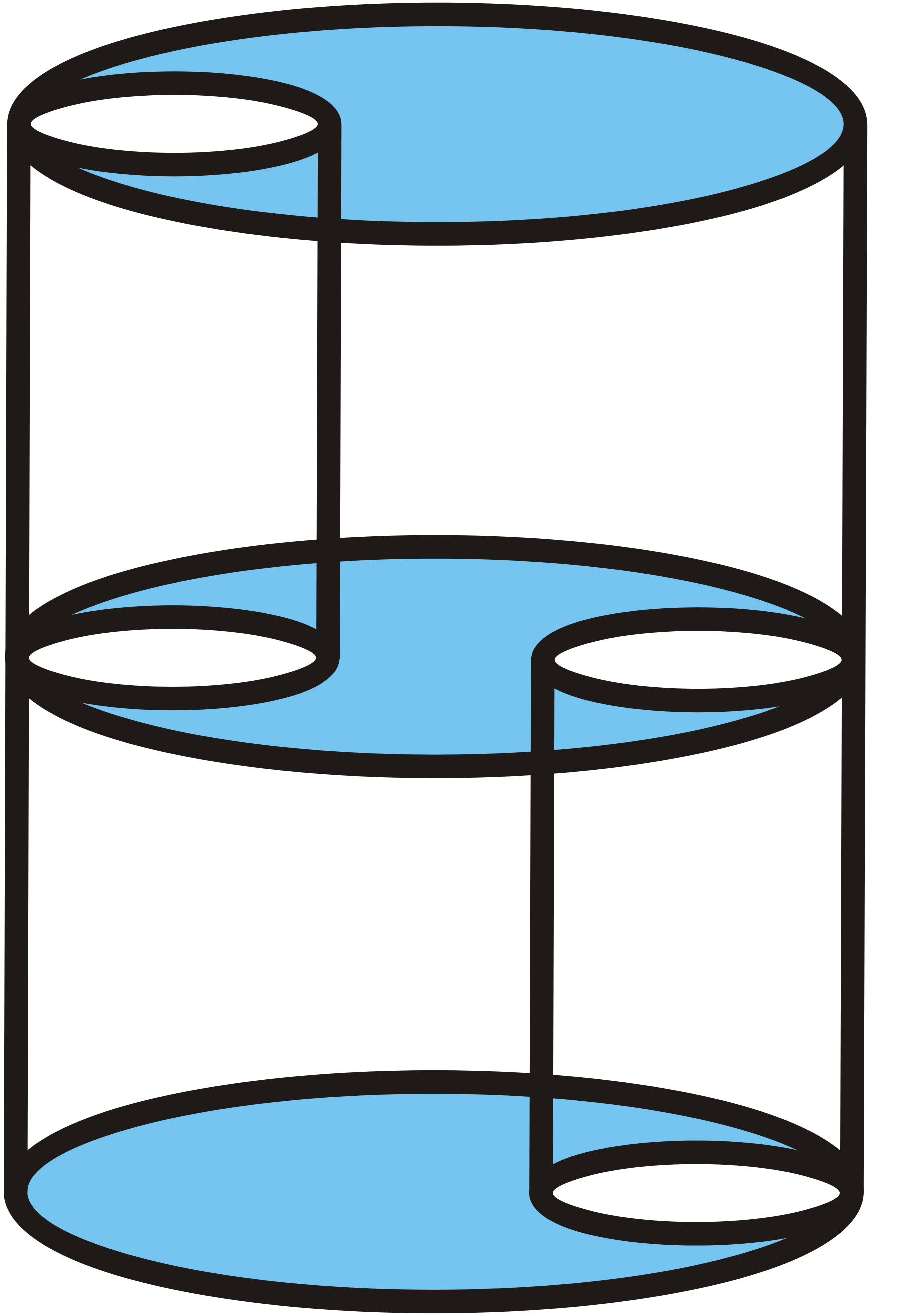


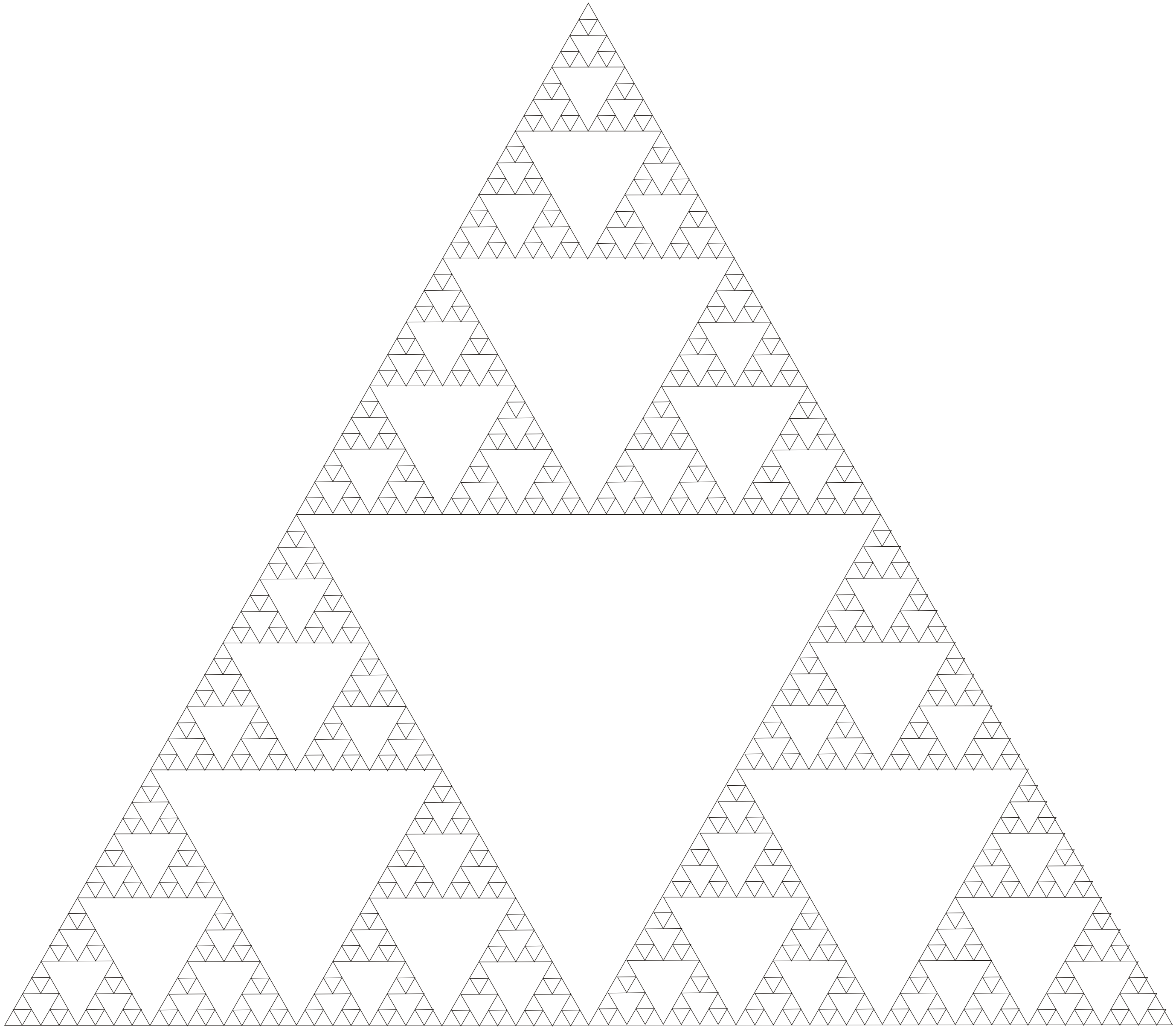


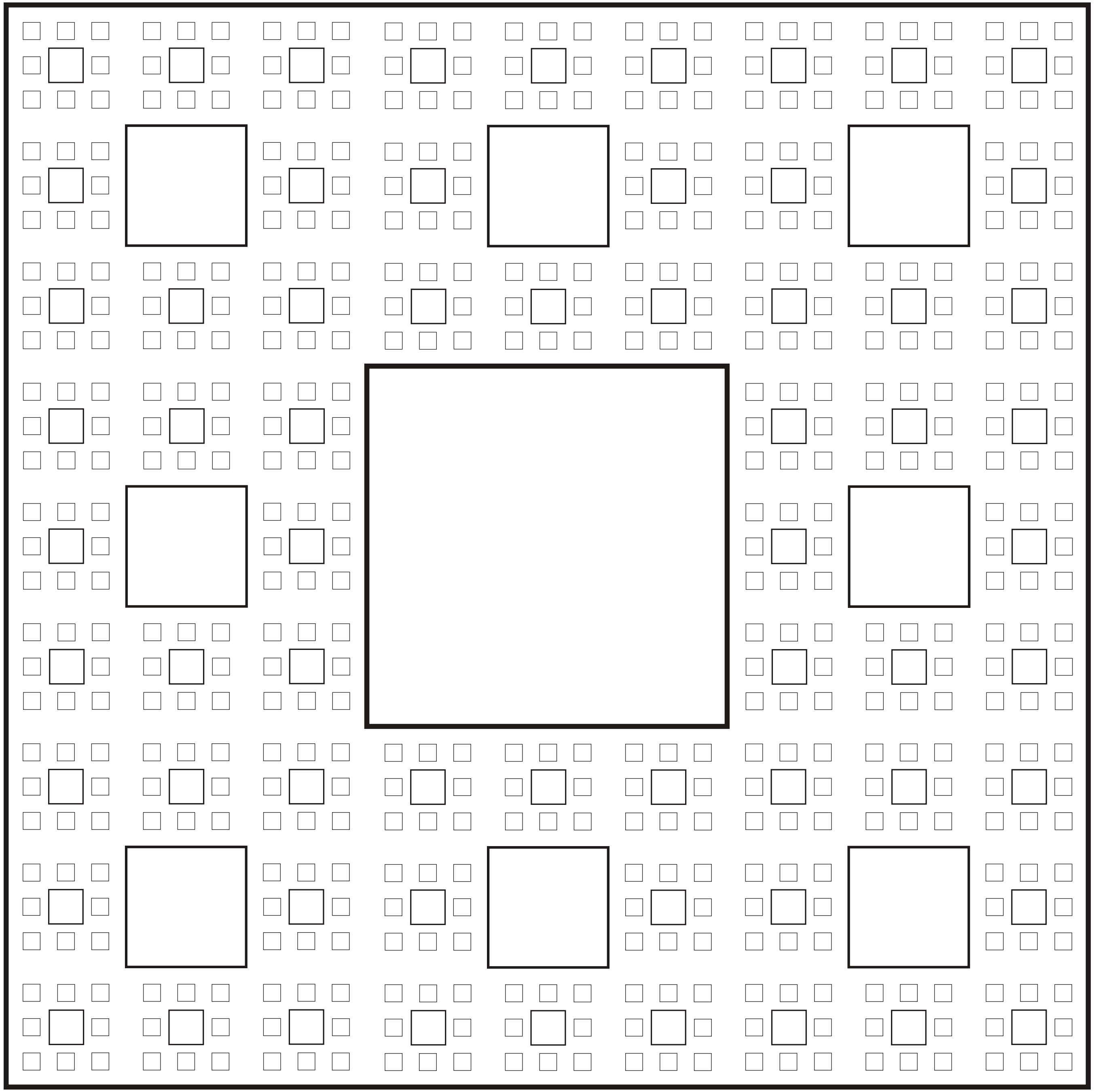


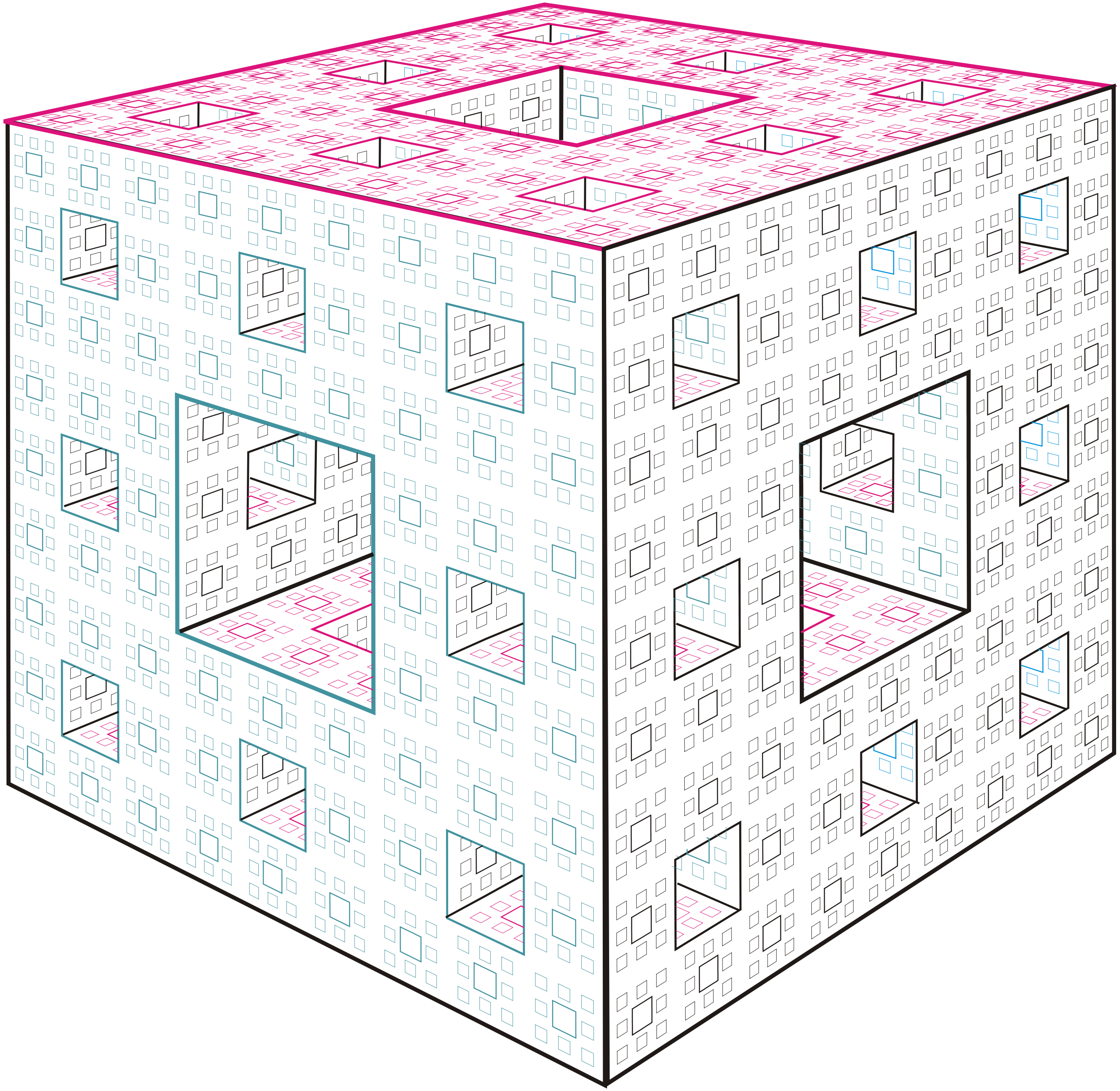


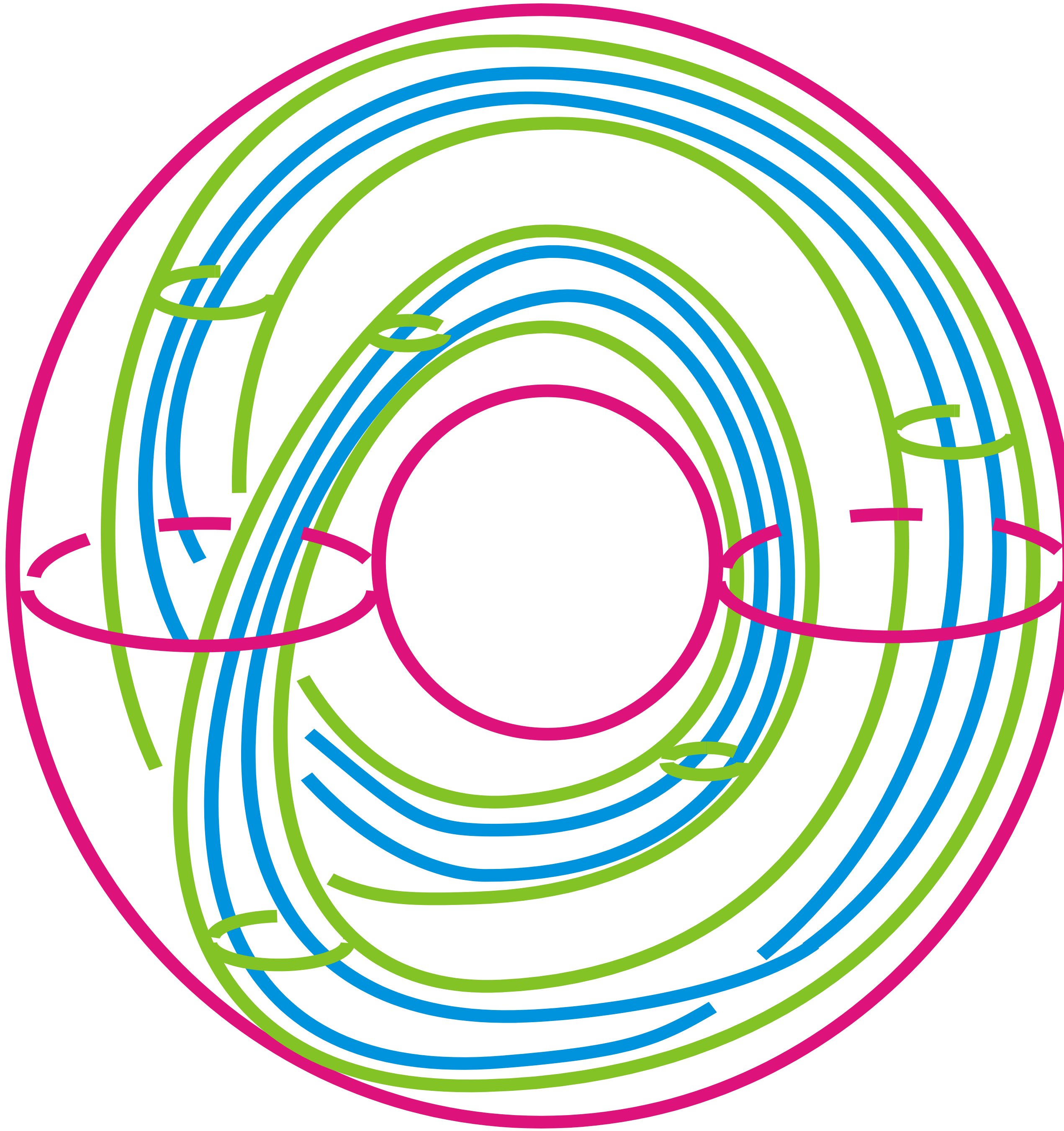


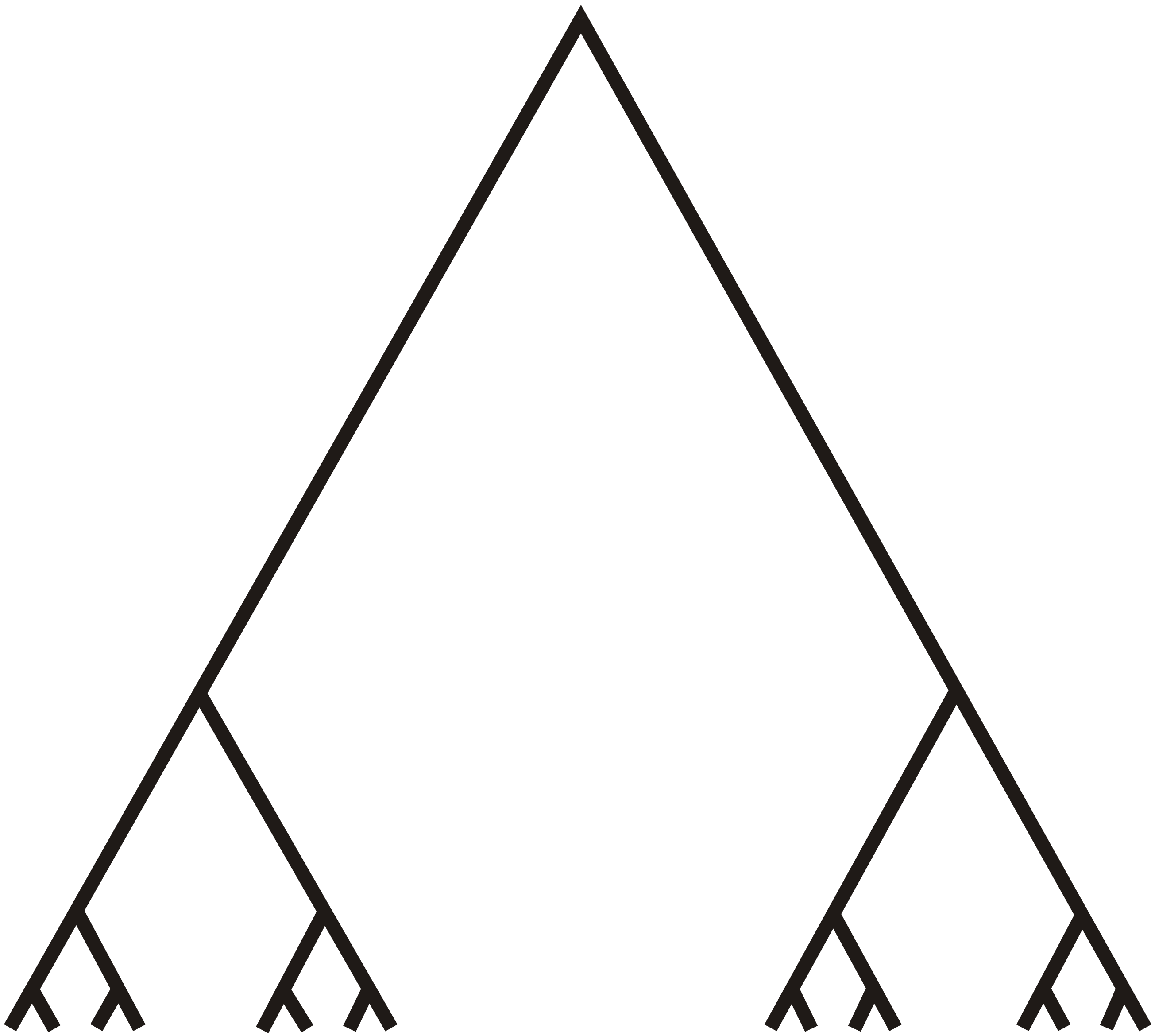


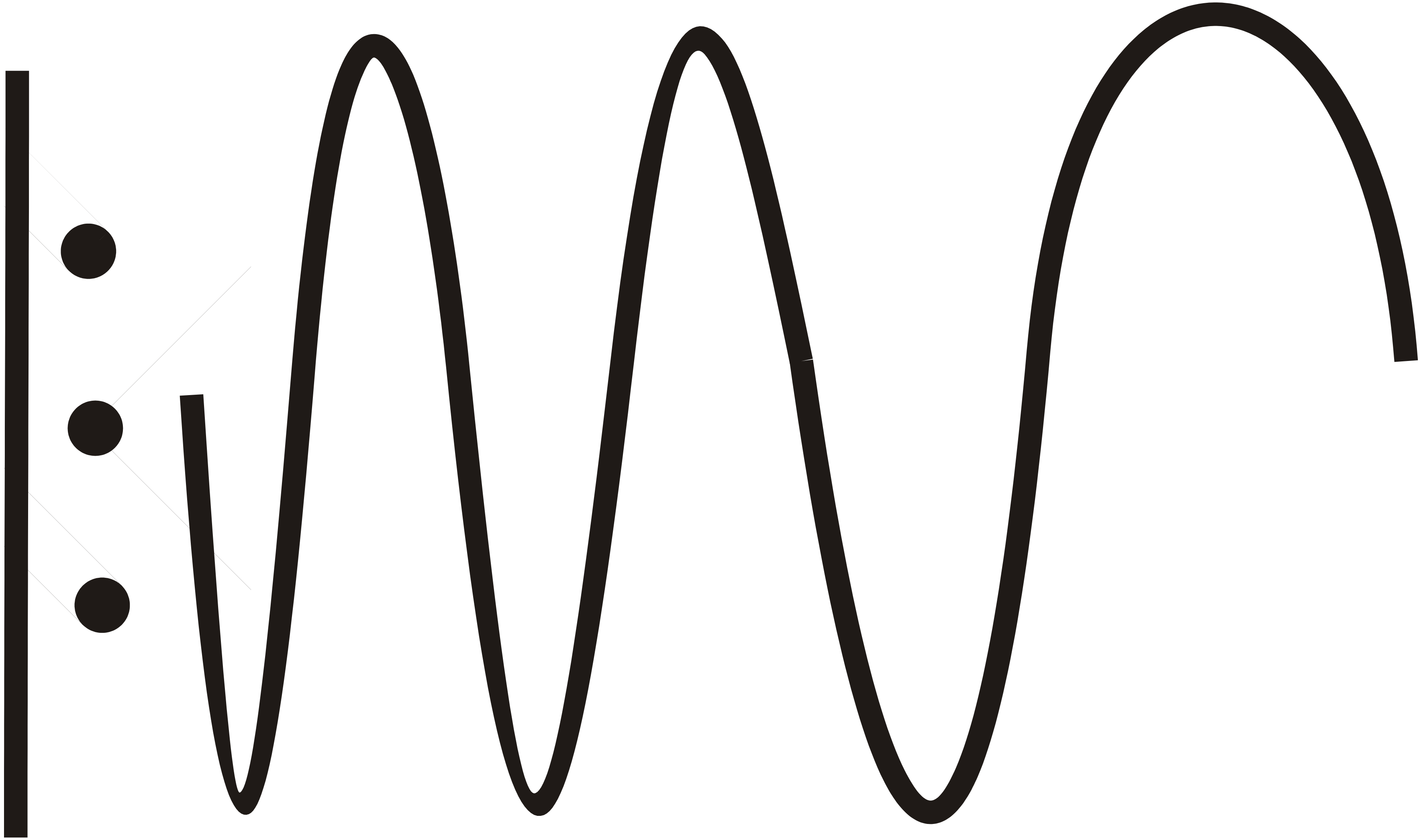


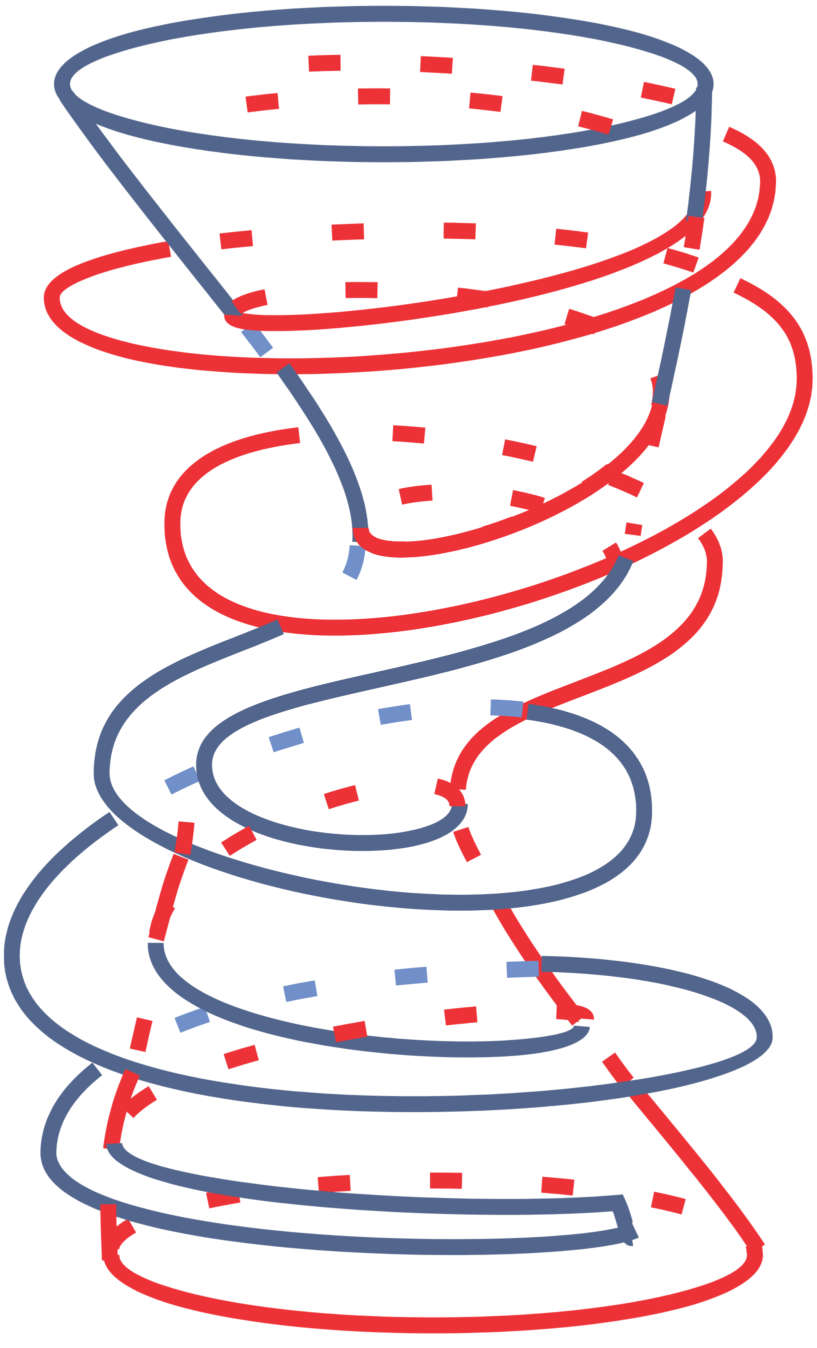








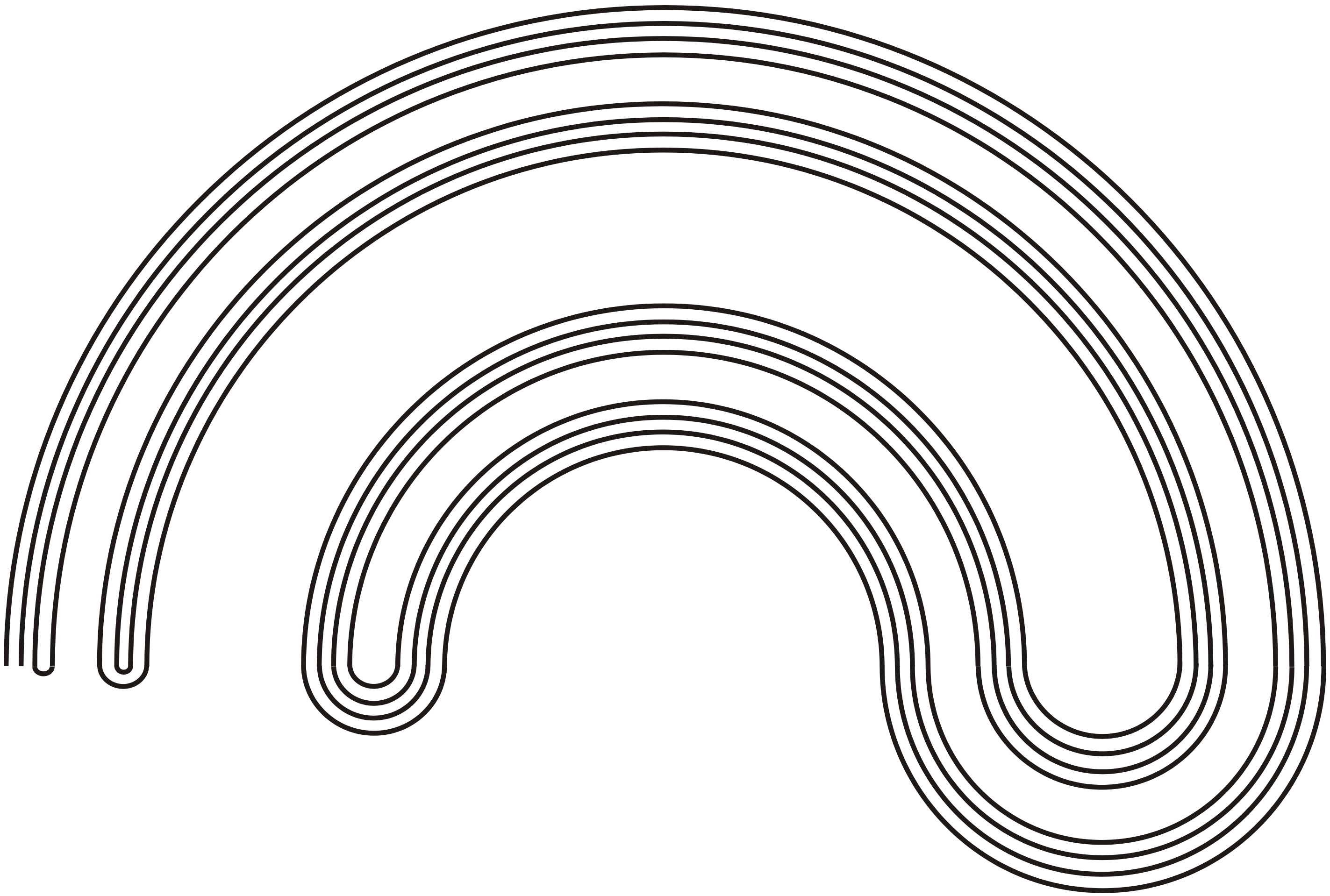




A continuum Y is ***indecomposable*** if Y is not the the union of two proper subcontinua.

A continuum X is ***hereditarily indecomposable*** if each of its subcontinua is indecomposable.

# Brouwer-Janiszewski-Knaster



Knaster and Kuratowski asked if there exists a hereditarily indecomposable continuum.

Knaster produced one in 1922.

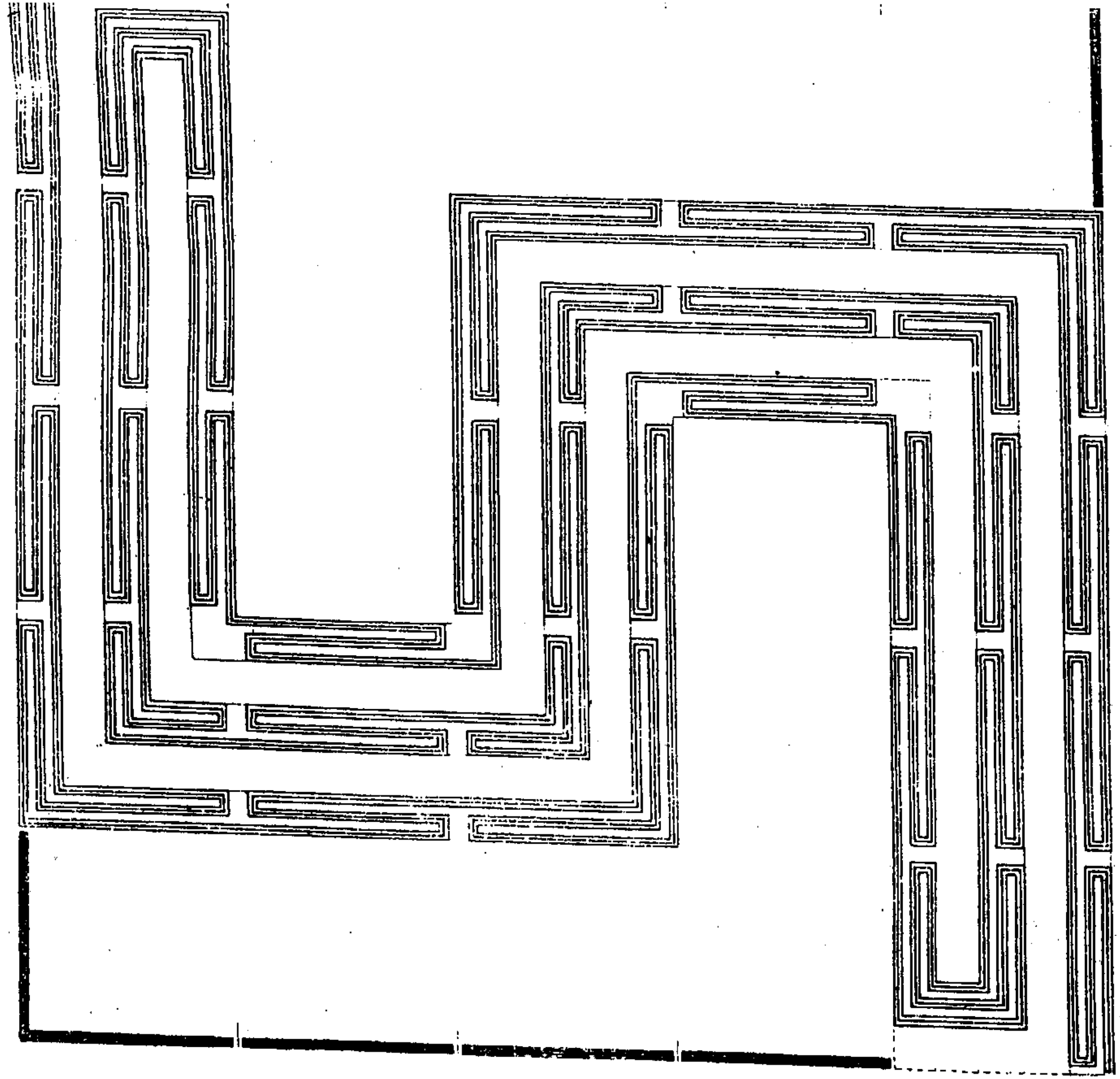


Fig. IV

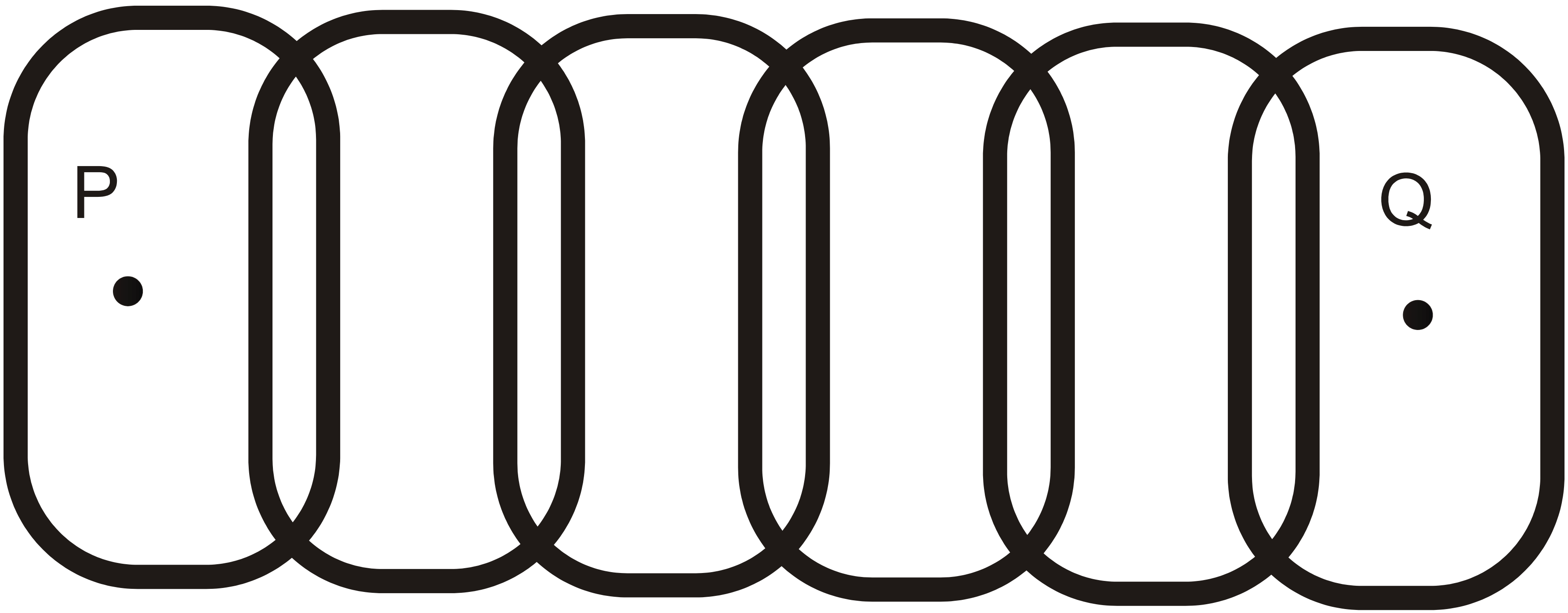
A continuum X is ***chainable*** if for each $r > 0$ there exists an open cover

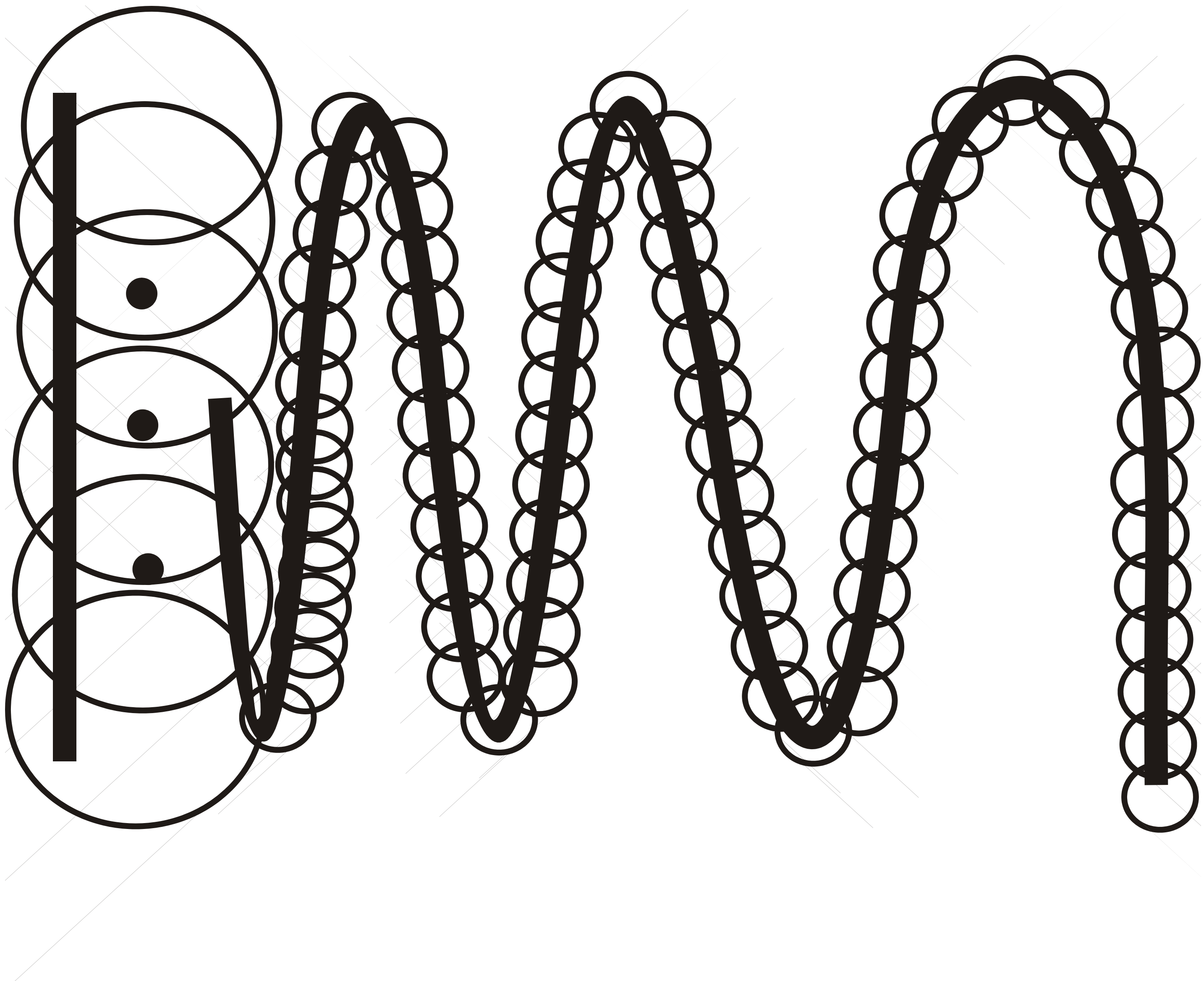
$$U_1, \ldots, U_n \text{ of } X \text{ s. t.}$$

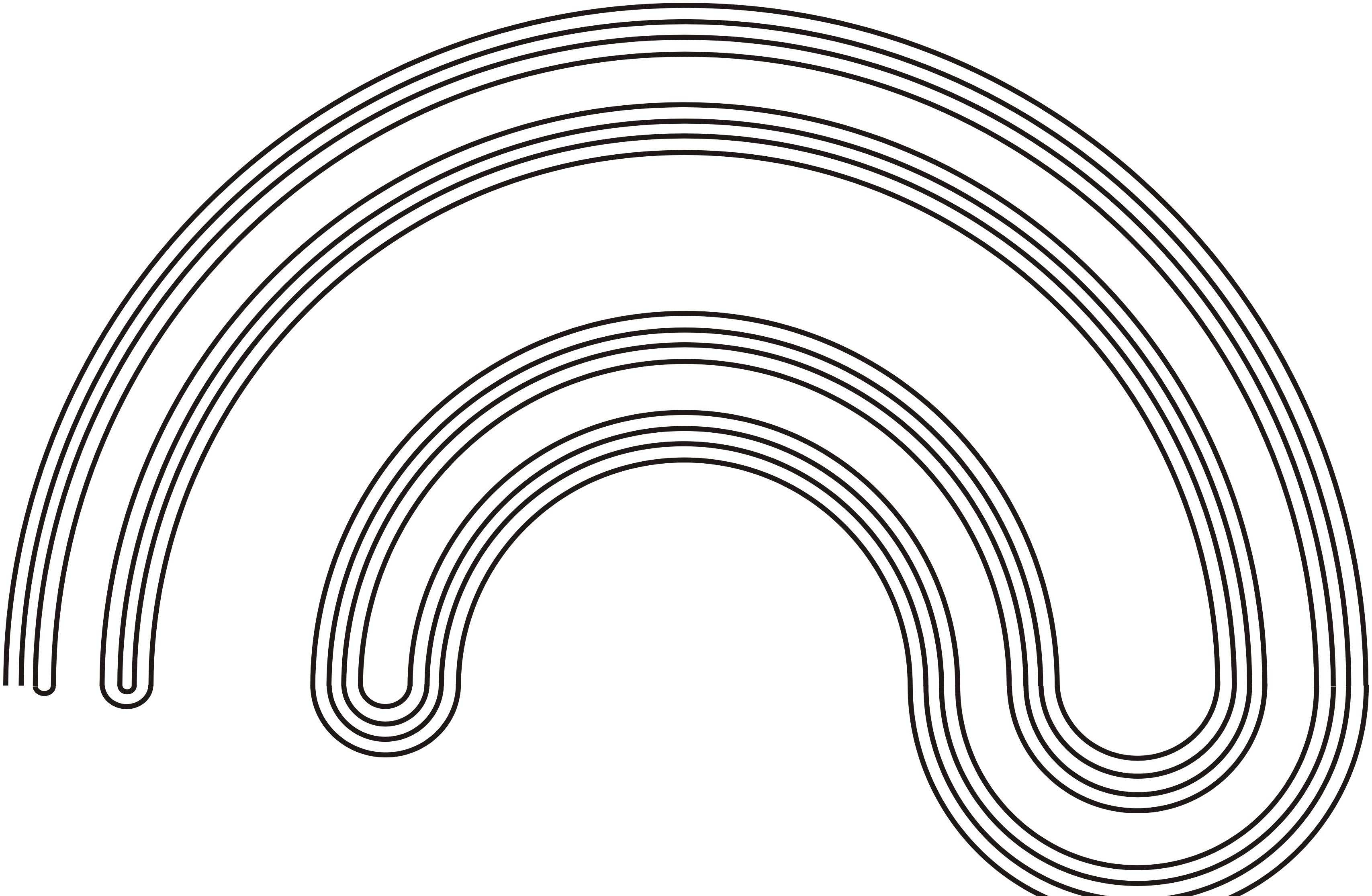
$$U_i \cap U_k \neq \phi$$

if and only if $|i - j| \leq 1$

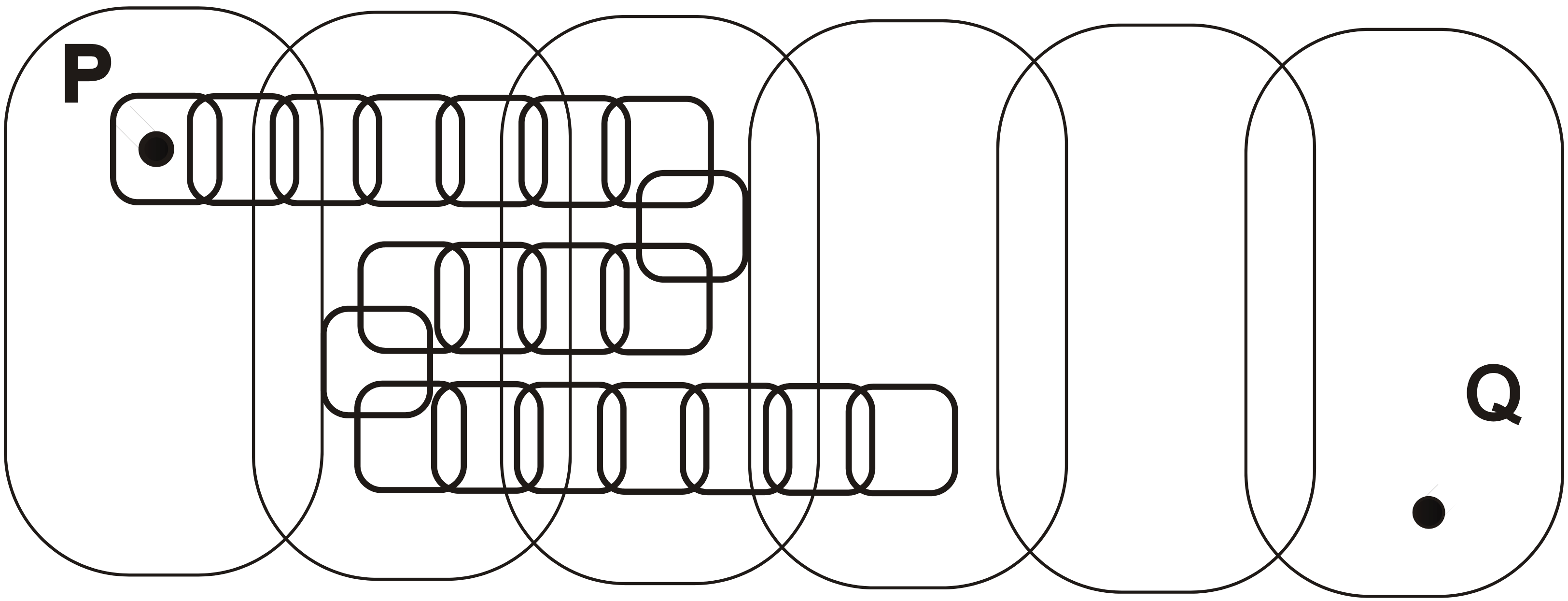
& each $U_i$ has diameter less than r.


P
Q





To go from one link $U_i$ to another link $U_j$ , with $i + 2 < j$, one has to visit first $U_{j-1}$ , then visit $U_{i+1}$ and then we can go to $U_j$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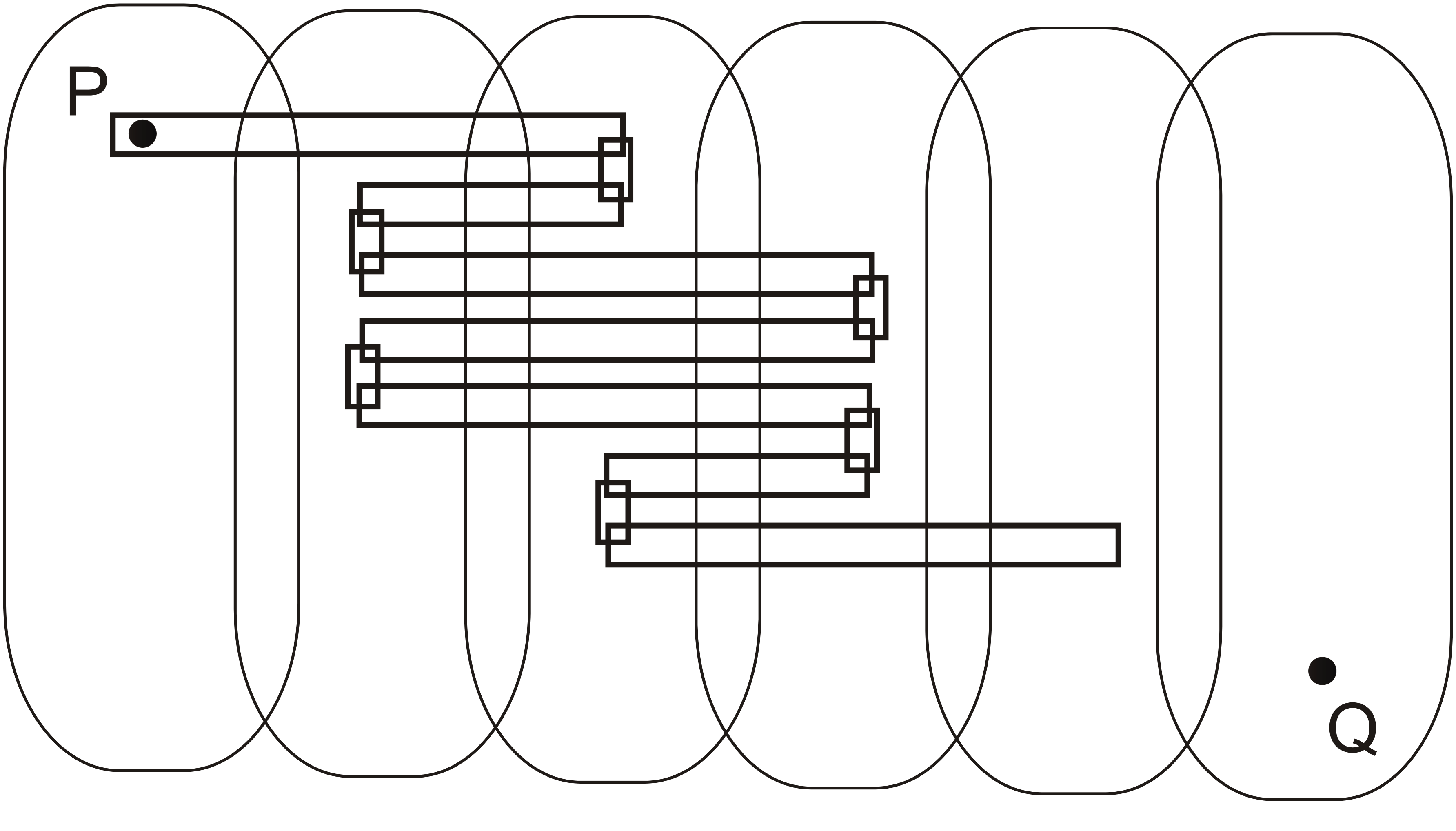
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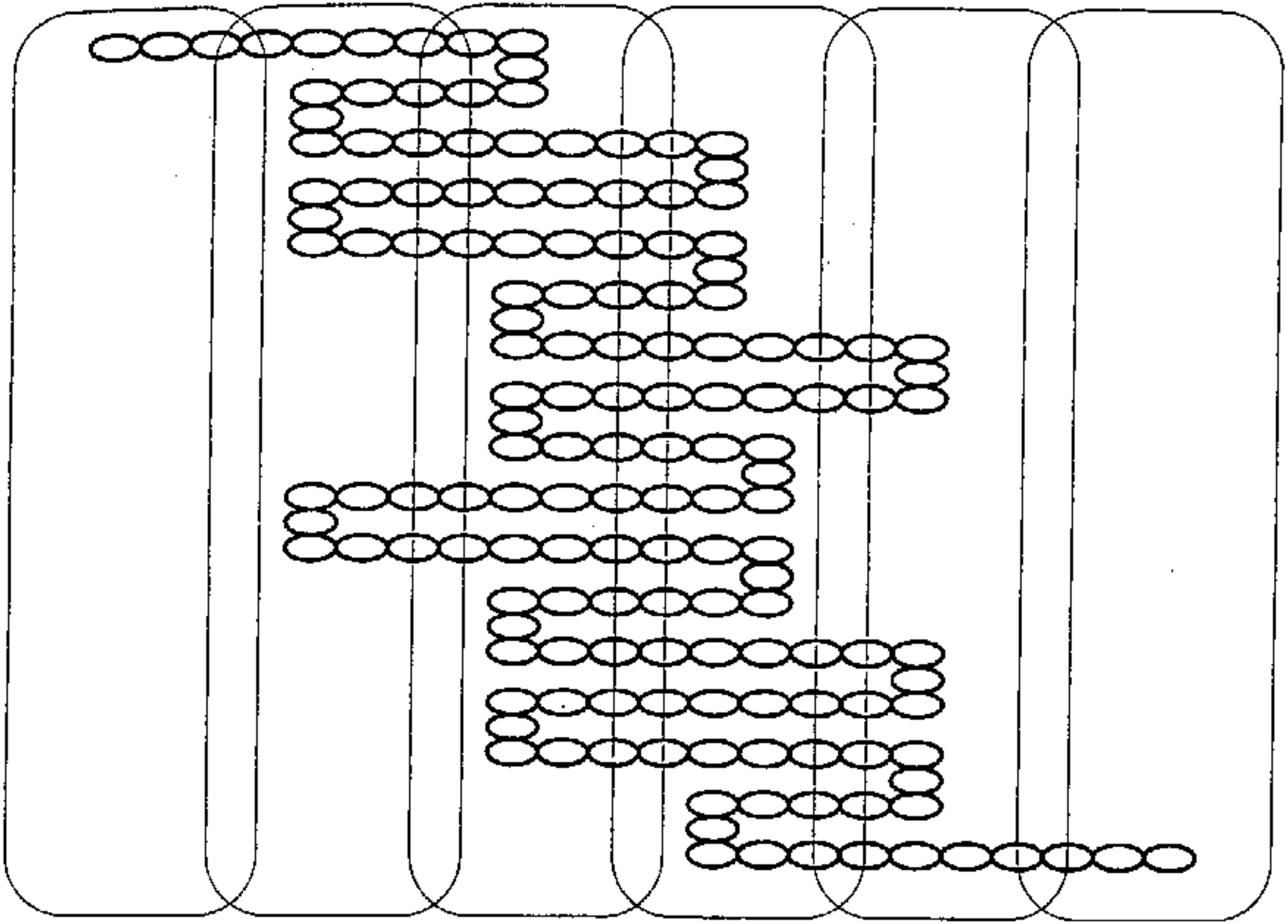
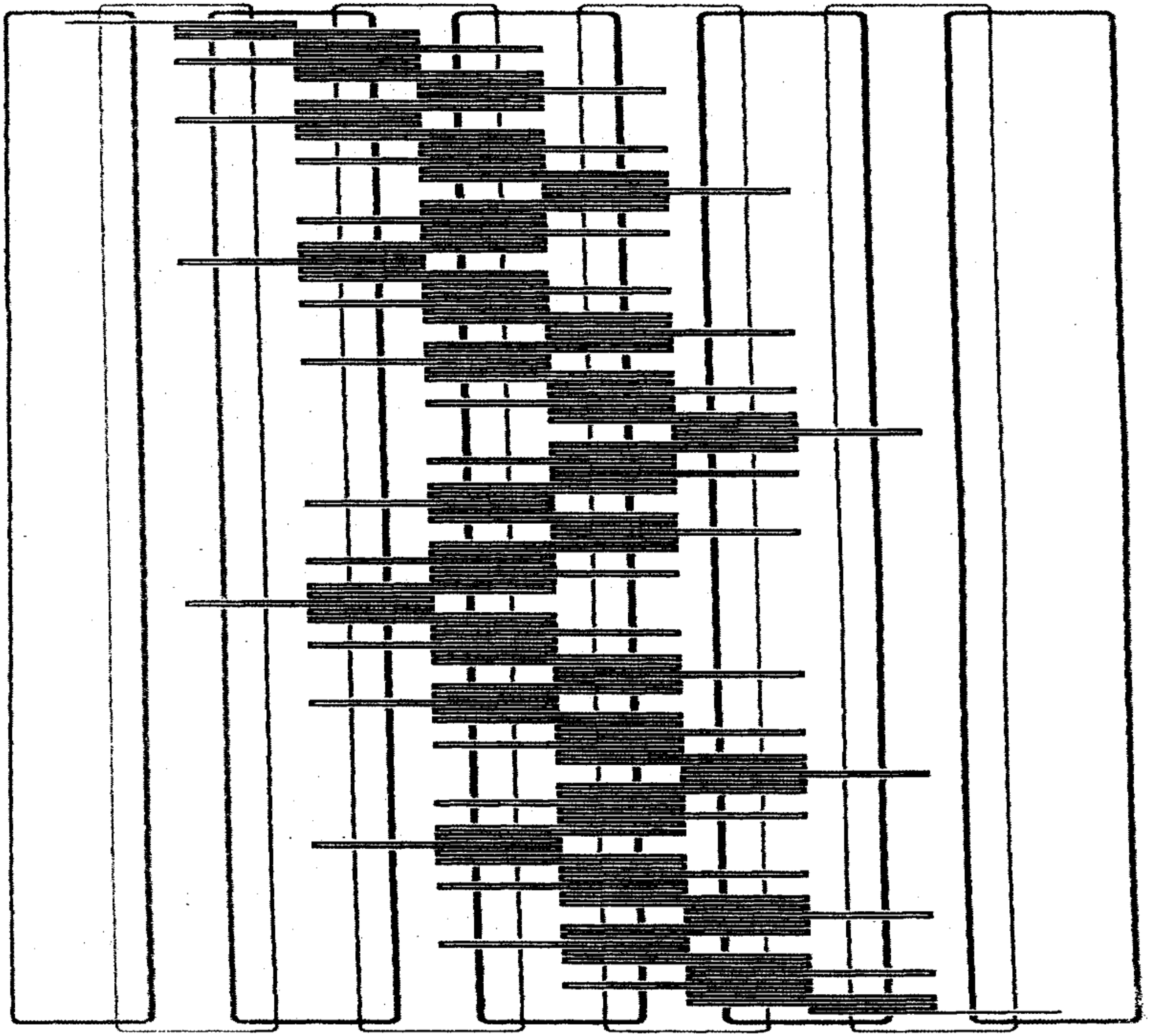




For each n,

$X_n = \cup$ links of the $n^{th}$-chain

$\mathbf{P} = X_1 \cap X_2 \cap \ldots$

**P** is the pseudo-arc.

**Theorem** (RH Bing, 1948)**.**
The pseudo-arc is homogeneous (if p,q are in P, then there exists a homeomorphism $h : P \to P$ such that $h(p) = q$).

**Theorem** (E. E. Moise, 1948)**.**
The pseudo-arc is homeomorphic to each of its nondegenerate subcontinua.

**Theorem** (RH Bing)**.**
The pseudo-arc is the only chainable hereditarily indecomposable continuum.

Issac Kapuano, 1953 published a “proof” that the pseudo-arc is not homogeneous.

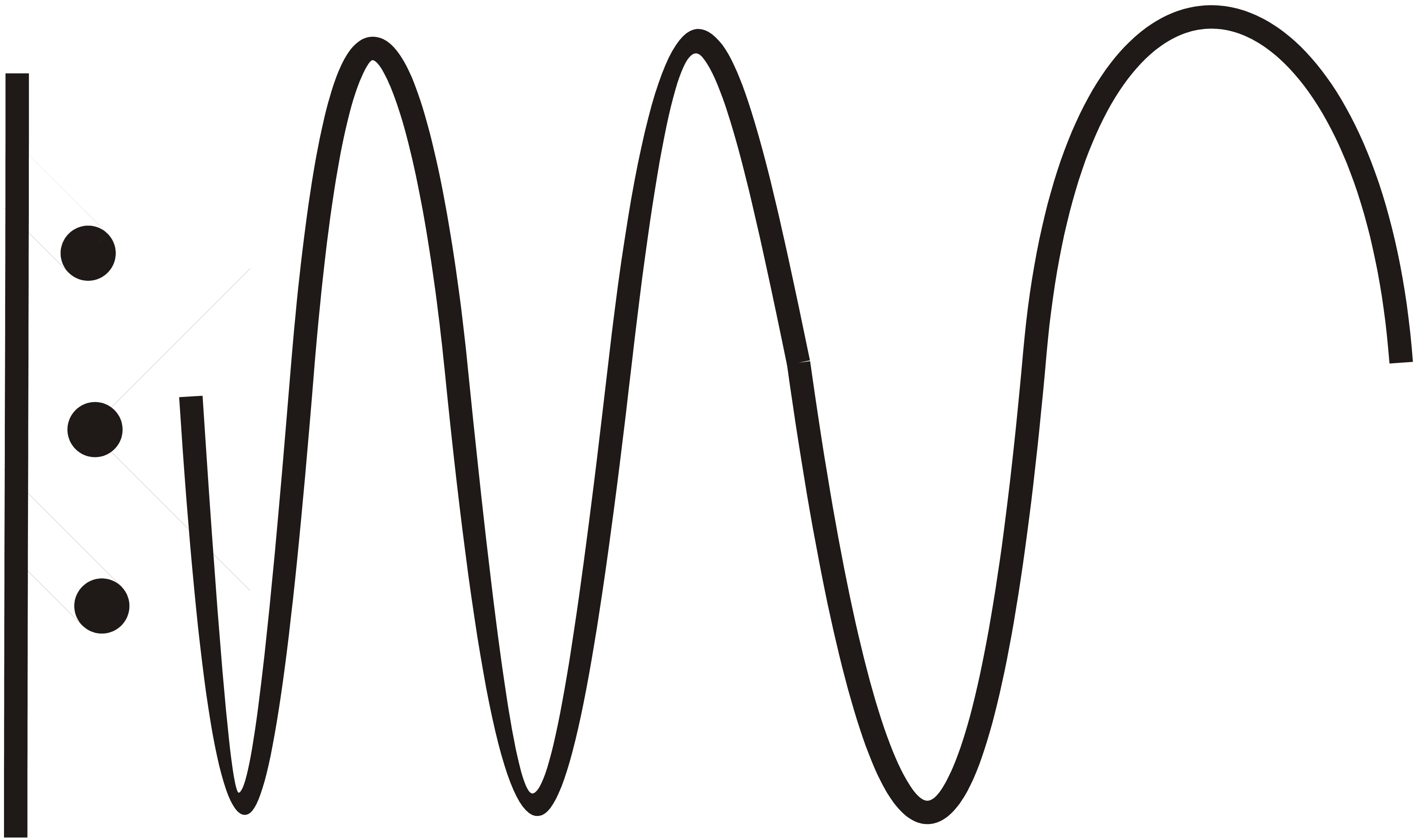
A. S. Evening-Volpin, in Referativnyl Zhurnal commented: “in the light of this, the problem of Knaster and Kuratowski remains open”

A. Lelek and M Rochowski produced a 60 pages monograph, with all the details,

**Problem 1.** Is there another continuum, different from P and [0,1] homeomorphic to each of its nondegenerate subcontinuum?

**Theorem** (L. Hoehn, and L. Oversteegen, 2014). There are only three homogeneous continua in the plane:
(a) a simple closed curve,
(b) the pseudo-arc,
(c) the circle of pseudo-arcs.

# 1. Compactifications of the ray [0,1)



Two continua X and Y are ***comparable*** provided that there exists an onto mapping from one to the other.

**Theorem** (W. Awartani, 1993)**.** There is an uncountable family, F, of non-mutually comparable compactifications of the ray [0,1), with remainder an arc.

**Theorem** (V. Martínez de la Vega, 2004)**.** There is an uncountable family of non-mutually homeomorphic compactifications of the ray [0,1), with remainder a pseudo-arc.

**Theorem** (V. Martínez de la Vega and P. Minc, 2015)**.** For each continuum X, there is an uncountable family of non-mutually homeomorphic compactifications of the ray [0,1), with remainder X.

**Theorem** (A. Illanes, P. Minc and F. Sturm, 2015)**.** If X and Y are compactifications of the ray [0,1), with remainder pseudo-arcs P and Q, respectively, then each onto mapping from P to Q can be extended to an onto mapping from X to Y.

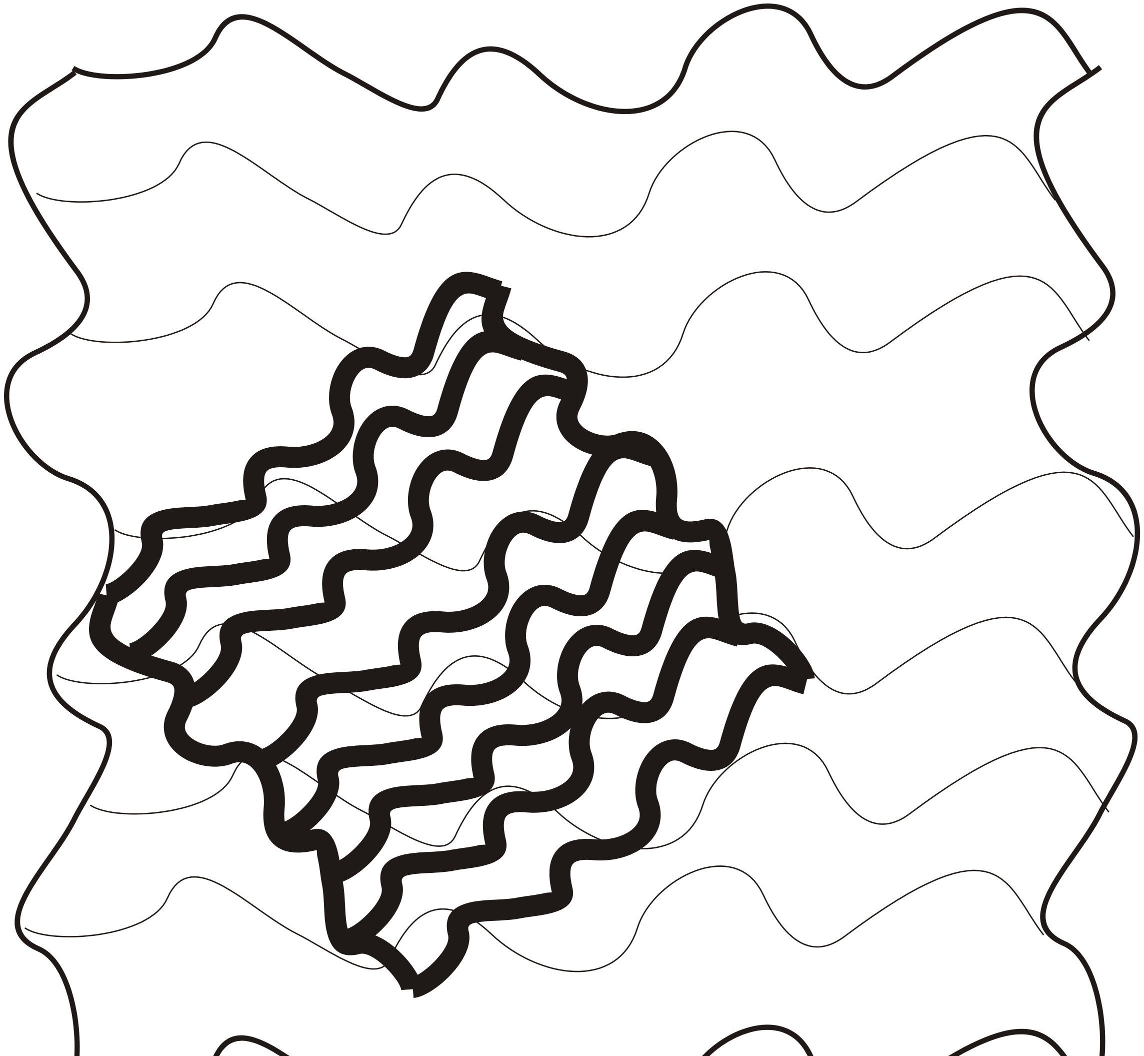
**Problem 2.** Is there a compactification of the ray [0,1), X, with remainder the pseudo-arc P such that every homeomorphism from P onto P can be extended to X?

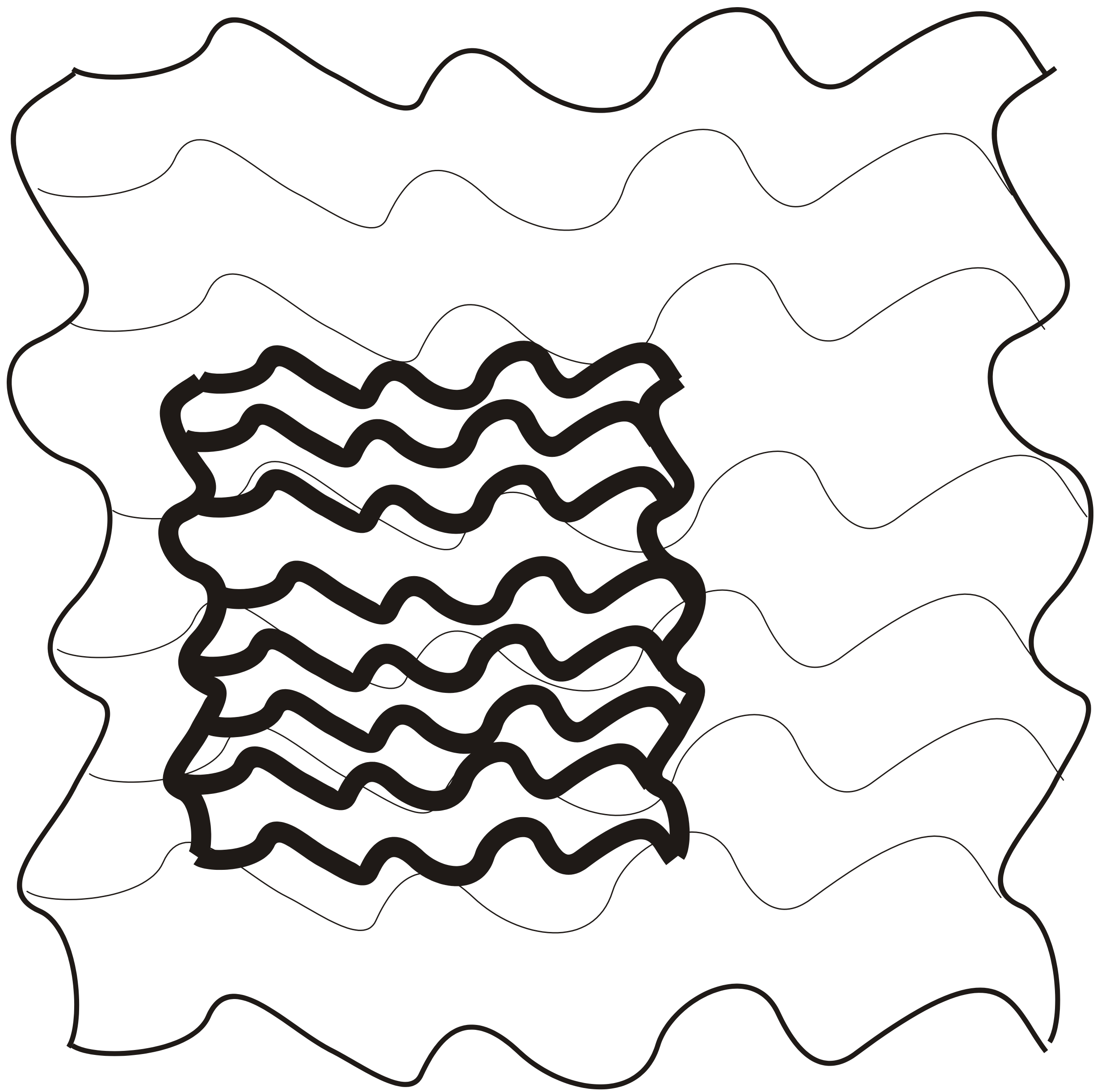
# 2. Embedding a product of pseudo-arcs into a product of pseudo-arcs

**Theorem** (D. P. Bellamy and J. Kennedy, 1986).
Let X be a finite or countable product of pseudo-arcs. Then each homeomorphism from X onto X is a product of homeomorphisms (up to permutation of coordinates).

**Theorem** (M. E. Chacón, A. Illanes and R. Leonel, 2012)**.**

If E : P x P $\rightarrow$ P x P is an embedding, then E is a product of embeddings, up to a permutation of coordinates.





**Problem 3.** If $E : P \times P \times P \to P \times P \times P$ is an embedding, then is E a product of embeddings, up to a permutation of coordinates?

What about embeddings of $P^n$ into $P^n$, for $n \geq 4$?

For a continuum X, the ***$n^{th}$-symmetric product*** of X is the hyperspace

$F_n(X)$ = { A c X : A is nonempty and contains at most n points}.

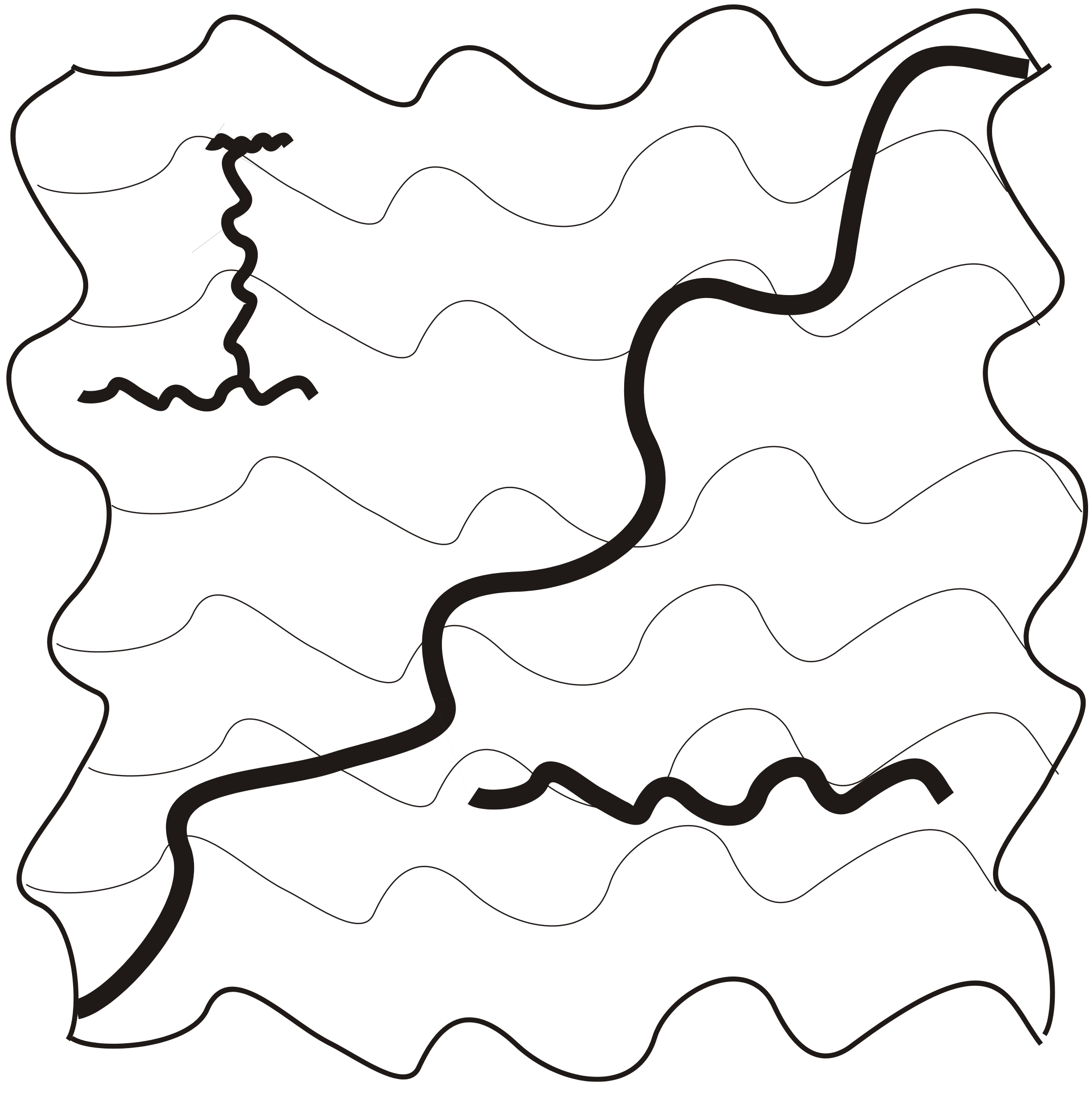
$F_n(X)$ is considered with the Hausdorff metric.

**Theorem** (I. Calderón, R. Hernández and A. Illanes, 2015).
If $E : F_2(P) \to F_2(P)$, is an embedding, then there is an embedding $e : P \to P$ such that $E(\{p,q\}) = \{e(p),e(q)\}$ for every $p,q \in P$.

**Corollary** (I. Calderón, R. Hernández and A. Illanes, 2015).
If $H : F_2(P) \to F_2(P)$, is a homeomorphism, then $H(F_1(P)) \subset F_1(P)$.

**Problem 4.** What about embeddings of $F_n(P)$ into $F_n(P)$, for $n \geq 3$?

# 3. The product P x P



**Problem** (David P. Bellamy, 2007).
Does each nondegenerate subcontinuum of $P^n$ contain a pseudo-arc?

**Example** (A.I., 2014)**.** There exists a subcontinuum of P x P that contains no pseudo-arcs.

# 6. Compact subsets of Euclidean spaces contained in pseudo-arcs

# 4. Compact subsets of Euclidean spaces contained in pseudo-arcs.

**Theorem** (R. L. Moore, J. R. Kline, 1919). In the plane, a closed and compact set M is a subset of an arc if and only if every component of M is either a one-point set or an arc α such that no point of α, except its end points, is a limit point of M - α.

**Theorem** (H. Cook, 1961)
If K is a compact plane set, then there exists a pseudo-arc P with $K \subset P \subset R^2$ if and only if each one of the nondegenerate components of K is a pseudo-arc.

**Theorem** (A. I., 2014)**.**
If $k \geq 3$ and $K$ is a compact subset of the Euclidean space $R^k$, then there exists a pseudo-arc $P$ such that $K \subset P \subset R^k$ if and only if each nondegenerate component of $K$ is a pseudo-arc.

**Problem** (D. Bellamy, 2007)**.** Let X be a compact subset of $R^k$ such that each of its nondegenerate components is hereditarily indecomposable continuum, does there exist a hereditarily indecomposable continuum X in $R^{k+1}$ such that $K \subset X$?

**Theorem** (A. I., 2014)**.** Let K be a compact subset of $R^k$ $(k > 2)$ such that K does not separate $R^k$ and each of its nondegenerate components is hereditarily indecomposable, then there exists a hereditarily indecomposable continuum X in $R^k$ such that $K \subset X$

# 5. Pseudo-arcs in the hyperspace of subcontinua of the pseudo-arc

Given a continuum X, C(X) denotes the hyperspace of subcontinua of X, with the Hausdorff metric

A Whitney map is a continuous function g from C(X) into [0,1] such that:

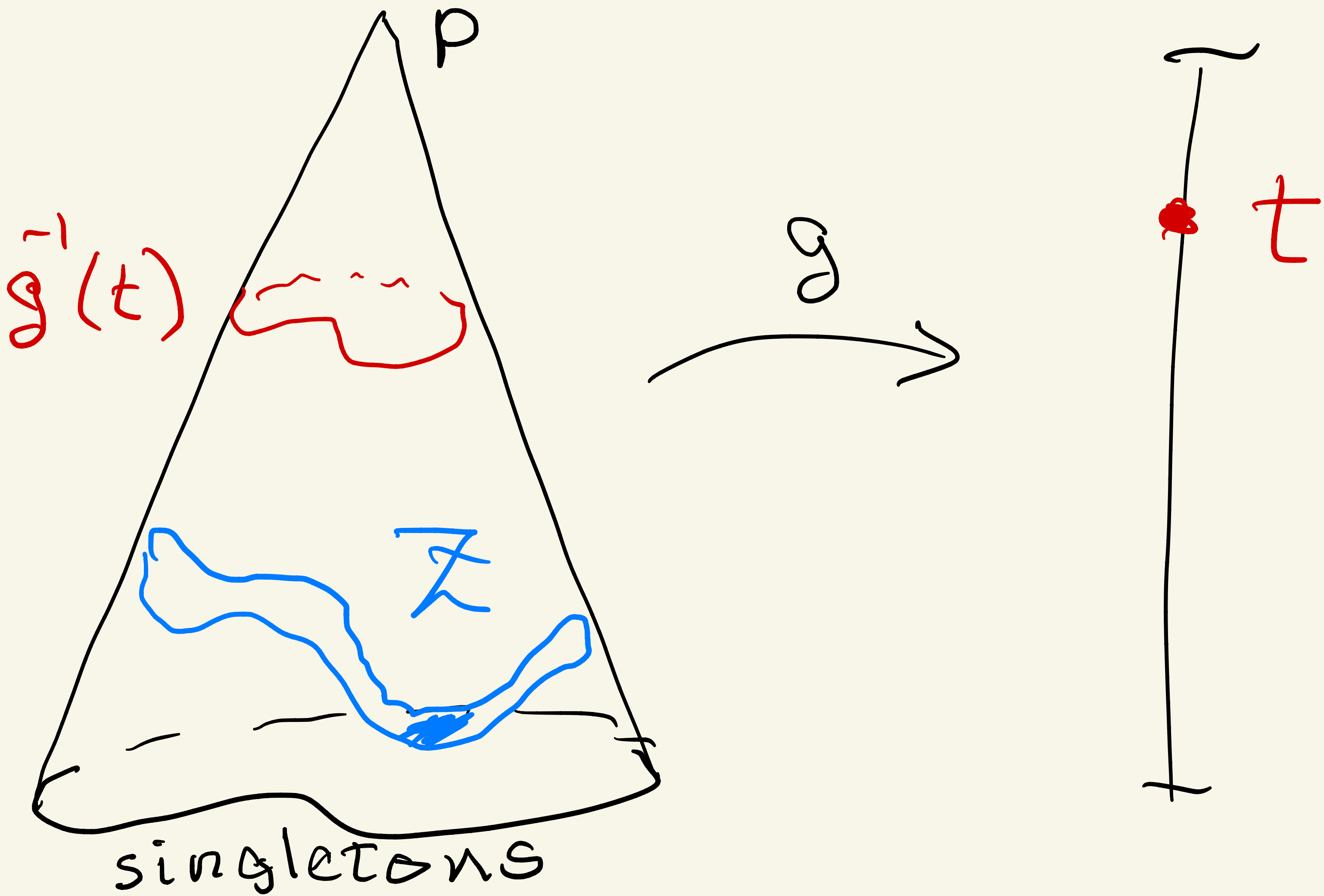
(a). $g(\{p\}) = 0$ for each p in X, and
(b). $g(A) < g(B)$, if A is a proper subcontinuum of B.

Whitney levels are the fibers of the Whitney maps.

Problem (Norman Passmore, 1976; David P. Bellamy, 2007).

Let P be the pseudo-arc and let Z be a pseudo-arc contained in C(P). Does Z is a subset of a Whitney level for some Whitney map?

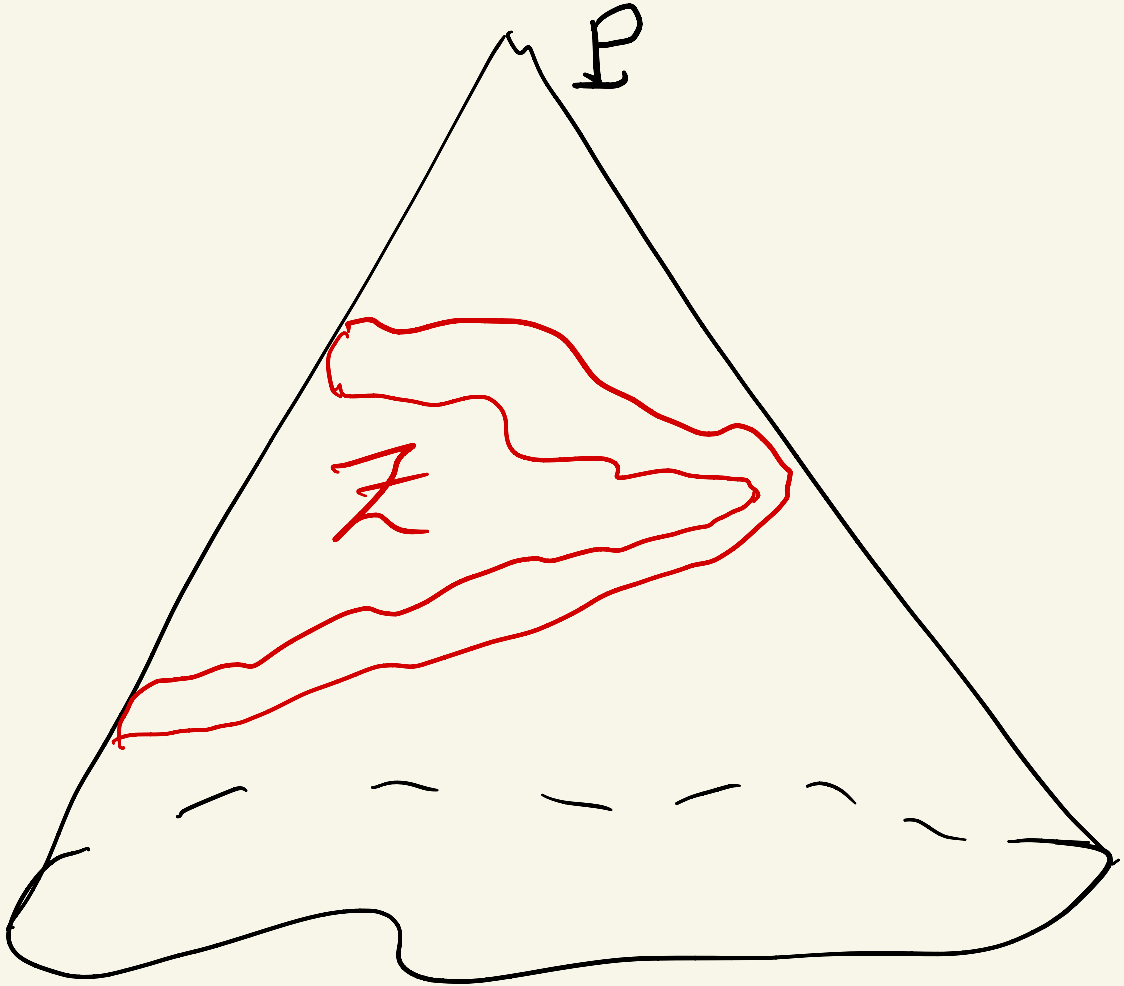
What if Z does not contain singletons?


p
$g^{-1}(t)$
$\mathcal{Z}$
singletons
g
t

Example (A. I., E. R. Marquez and J. M. Martinez-Montejano, 2022).

-There exists a pseudo-arc Z in C(P) such that Z does not contain singletons and Z is not contained in any Whitney level.

-There exists a pseudo-arc contained in the cone of P such that Z does not have the vertex and its projection on P is not one-to-one.



P
Z

Thanks