Hyperspaces of Erdős spaces

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13th Prague Symposium on General Topology
Prague, Czech Republic, July 25, 2022

Introduction to Erdős spaces

Hyperspaces

Characterization of $\mathbb{Q} \times \mathfrak{E}_c$

Questions

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Erdős spaces

Let ℓ_2 be the set of square-summable sequences of reals.

$$\begin{aligned}\mathfrak{E} &= \{(x_n)_{n\in\omega} \in \ell_2 \colon \forall n \in \omega,\ x_n \in \mathbb{Q}\} \\ \mathfrak{E}_c &= \{(x_n)_{n\in\omega} \in \ell_2 \colon \forall n \in \omega,\ x_n \in \{0\} \cup \{1/n \colon n \in \mathbb{N}\}\}\end{aligned}$$

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Theorem (Paul Erdős, 1940)

If $E \in \{\mathfrak{E}, \mathfrak{E}_c\}$ *then* $dim(E) = dim(E^2) = 1$.

Topology in ℓ_2

Lemma (folklore)

Let $(x(n))_{n\in\omega}$ be a sequence in ℓ_2 and $x \in \ell_2$. Then the following two conditions are equivalent:

(a) $x = \lim_{n\to\infty} x(n)$, *and*

(b) (i) *for each* $i \in \omega$, $x_i = \lim_{n\to\infty} x(n)_i$, *y*
 (ii) $\|x\| = \lim_{n\to\infty} \|x(n)\|$.

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Observation

Closed ℓ_2-balls are closed in the product topology of $\mathbb{R}^{\mathbb{N}}$.

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Almost zero dimensional

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Notice that the product topology on $\mathfrak{E}$ and $\mathfrak{E}_c$ is zero dimensional.

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Definition

A separable metric space $\langle X, \tau\rangle$ is **almost zero dimensional** *(AZD) if there is a topology $\mathcal{W}$ on X such that $\mathcal{W} \subset \tau$, $\langle X, \mathcal{W}\rangle$ is zero dimensional and there is a basis of neighborhoods of $\langle X, \tau\rangle$ that are $\mathcal{W}$-closed.*

Almost zero dimensional

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Alternative definition

Definition

A C-**set** *in a topological space X is a set that is an intersection of clopen sets.*

Lemma

A (separable metrizable) space is AZD if and only if it has a basis consisting of C-sets.

Properties of almost zero dimensional spaces

- Every zero-dimensional space is AZD.

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- Every AZD space is totally disconnected.
- Every AZD compact space is zero-dimensional.
- Every subspace of an AZD space is AZD.
- The (countable) product of AZD spaces is AZD.

Cohesion

Definition

A space X is **cohesive** *if for every point $p \in X$ there is an open set U with $p \in U$ and U does not contain clopen non-empty subsets.*

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Theorem (Erdős, 1940)

$\mathfrak{E}$ and $\mathfrak{E}_c$ are cohesive.

Motivation: characterizations of zero-dimensional, separable metric spaces

- (Brouwer, 1910) 2^ω: compact; nowhere countable.
- (Sierpinski, 1920) $\mathbb{Q}$: countable; no isolated points.
- (Alexandroff and Urysohn, 1928) ω^ω: G_δ; nowhere locally compact.
- (Alexandroff and Urysohn, 1928) $\mathbb{Q} \times 2^\omega$: K_σ; nowhere locally compact; nowhere countable.
- (van Mill, 1981) $\mathbb{Q} \times \omega^\omega$: $G_{\delta\sigma}$; nowhere G_δ; nowhere K_σ.
- (Van Engelen, 1986) ω_1-many Borel spaces.

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“Intrinsic” characterization of $\mathfrak{E}_c$

Theorem (Dijkstra and van Mill, 2009)

A separable metric space E is homeomorphic to $\mathfrak{E}_c$ if and only if E is cohesive and there is a witness topology $\mathcal{W}$ and a basis of neighborhoods of E that are **compact** *in $\mathcal{W}$.*

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Theorem (Dijkstra, van Mill and Steprāns, 2004)

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Theorem (Dijkstra, van Mill and Steprāns, 2004)

$\mathfrak{E}_c \not\approx \mathfrak{E}_c^\omega$.

$\mathfrak{E}_c^\omega$ is called **stable Erdős space**.

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Ph.D. project: study Vietoris hyperspaces of Erdős spaces

Figure: My PhD student, Alfredo Zaragoza (2019 picture).

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Vietoris topology

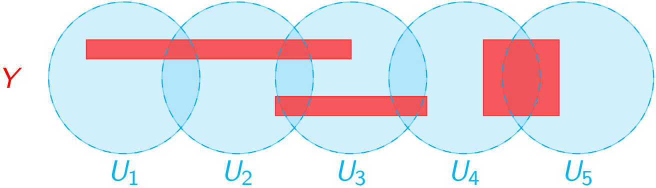
For a space X the hyperspace of closed sets is

$$CL(X) = \{A \subset X \colon A \text{ is closed and non-empty}\},$$

where a basic open set is

$$\langle U_1, \ldots, U_m \rangle = \{Y \in CL(X) \colon Y \subset U_1 \cup \ldots \cup U_n \wedge \forall i \leq m \, (Y \cap U_i \neq \emptyset)\},$$

and $U_1, \ldots, U_m$ are open.



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Vietoris hyperspaces

We will consider the following subspaces of $CL(X)$

$$\begin{aligned}
\mathcal{K}(X) &= \{A \in CL(X)\colon A \text{ is compact}\}, \\
\mathcal{F}(X) &= \{A \in CL(X)\colon A \text{ is finite}\}, \text{ and} \\
\mathcal{F}_n(X) &= \{A \in CL(X)\colon |A| = n\} \text{ for each } n \in \mathbb{N}.
\end{aligned}$$

$\mathcal{F}_n(X)$ is the **nth symmetric product** of X or **nth symmetric power** of X.

Theorem (E. Michael, 1950)

(a) If X is separable and metrizable, then $\mathcal{K}(X)$ is separable and metrizable.

(b) If X is Polish, then $\mathcal{K}(X)$ is Polish.

(c) If X is zero-dimensional, then $\mathcal{K}(X)$ is zero-dimensional.

Motivation: hyperspaces of zero-dimensional spaces

Theorem (folklore)

$\mathcal{K}(2^\omega) \approx 2^\omega$.

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Theorem (Shirota, 1968)

$\mathcal{K}(2^{\omega_1}) \approx 2^{\omega_1}$.

Theorem (Šhapiro, 1979)

If $\kappa \geq \omega_2$, $\mathcal{K}(2^\kappa) \not\approx 2^\kappa$.

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Problem for Alfredo

Let E be $\mathfrak{E}$ or $\mathfrak{E}_c$. Are $\mathcal{K}(E)$, $\mathcal{F}(E)$, $\mathcal{F}_n(E)$ $(n \in \mathbb{N})$ homeomorphic to E?

First results in hyperspaces

Theorem ([Zaragoza, 2020])

For a separable metrizable space X the following are equivalent.

(a) X is AZD,

(b) $\mathcal{K}(X)$ is AZD,

(c) for all $n \in \mathbb{N}$, $\mathcal{F}_n(X)$ is AZD, and

(d) there exists $n \in \mathbb{N}$ such that $\mathcal{F}_n(X)$ is AZD.

Theorem ([Zaragoza, 2020])

If X is separable, metrizable, cohesive and AZD then the hyperspaces $\mathcal{K}(X)$, $\mathcal{F}(X)$ and all symmetric powers of X are cohesive.

Theorem ([Zaragoza, 2020])

For every $n \in \mathbb{N}$, $\mathcal{F}_n(\mathfrak{E}) \approx \mathfrak{E}$ and $\mathcal{F}_n(\mathfrak{E}_c) \approx \mathfrak{E}_c$.

Responses

Theorem ([Lipham, 2021])

$\mathcal{F}(\mathfrak{E}) \times \mathfrak{E} \approx \mathfrak{E}$; *that is,* $\mathcal{F}(\mathfrak{E})$ *is an Erdős space factor.*

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The hyperspace of compact sets

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$$\mathcal{K}(\mathfrak{E}_c) \subsetneq \mathcal{K}(\mathfrak{E}_c, \mathcal{W}).$$

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Is $\mathcal{K}(\mathfrak{E}_c)$ *homeomorphic to* $\mathfrak{E}_c$ *or* $\mathfrak{E}_c^\omega$*?*

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$\mathfrak{E}$ contains a closed copy of $\mathbb{Q}$ so $\mathcal{K}(\mathfrak{E})$ contains a closed copy of $\mathcal{K}(\mathbb{Q})$.

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But it is known that $\mathcal{K}(\mathbb{Q})$ is a topological group.

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Question ([Zaragoza, 2022])

Is $\mathcal{K}(\mathfrak{E})$ homogeneous?

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What about $\mathcal{F}(\mathfrak{E}_c)$?

Notice that

$$\mathcal{F}(\mathfrak{E}_c) = \bigcup\{\mathcal{F}_n(\mathfrak{E}_c)\colon n \in \mathbb{N}\}$$

so $\mathcal{F}(\mathfrak{E}_c)$ is the **increasing** union of nowhere dense copies of $\mathfrak{E}_c$.

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Alfredo observed: $\mathbb{Q}$ is an **increasing** union of compact countable spaces

$$\mathbb{Q} = \bigcup\{K_n\colon n \in \mathbb{N}\}$$

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so

$$\mathbb{Q} \times \mathfrak{E}_c = \bigcup\{K_n \times \mathfrak{E}_c \colon n \in \mathbb{N}\},$$

and $K_n \times \mathfrak{E}_c \approx \mathfrak{E}_c$ for each $n \in \mathbb{N}$.

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so

$$\mathbb{Q} \times \mathfrak{E}_c = \bigcup\{K_n \times \mathfrak{E}_c\colon n \in \mathbb{N}\},$$

and $K_n \times \mathfrak{E}_c \approx \mathfrak{E}_c$ for each $n \in \mathbb{N}$.

Conjecture

$\mathcal{F}(\mathfrak{E}_c) \approx \mathbb{Q} \times \mathfrak{E}_c$

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Cohesion, ver. 2

Definition

A space X is **cohesive** *with respect a collection of subsets $\{X_s : s \in I\}$ if every point $p \in X$ has a neighborhood U such that for all $s \in I$, $U \cap X_s$ contains no non-empty clopen subsets of X_s.*

Class $\sigma\mathcal{E}$

Definition ([HG and Zaragoza, 2022])

$\sigma\mathcal{E}$ is the class of all separable metrizable spaces E such that there exists a topology $\mathcal{W}$ on E that is witness to the almost zero-dimensionality of E, a collection $\{E_n \colon n \in \omega\}$ of subsets of E and a basis β of neighborhoods of E such that

(a) $E = \bigcup\{E_n \colon n \in \omega\}$,

(b) *for each* $n \in \omega$, E_n *is a crowded nowhere dense subset of* E_{n+1},

(c) *for each* $n \in \omega$, E_n *is closed in* $\mathcal{W}$,

(d) E *is* $\{E_n \colon n \in \omega\}$*-cohesive, and*

(e) *for each* $V \in \beta$, $V \cap E_n$ *is compact in* $\mathcal{W} \restriction E_n$ *for each* $n \in \omega$.

Class $\sigma\mathcal{E}$ contains one space up to homeomorphism

Lemma ([HG and Zaragoza, 2022])

$\mathbb{Q} \times \mathfrak{E}_c \in \sigma\mathcal{E}$.

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$\mathcal{F}(\mathfrak{E}_c) \in \sigma\mathcal{E}$.

Theorem ([HG and Zaragoza, 2022])

For a separable metrizable space E the following are equivalent:

(a) $E \in \sigma\mathcal{E}$, *and*

(b) $E \approx \mathbb{Q} \times \mathfrak{E}_c$.

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Hyperspaces and $\mathbb{Q} \times \mathfrak{E}_c$

Corollary ([HG and Zaragoza, 2022])

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$\mathcal{F}(\mathfrak{E}_c) \approx \mathbb{Q} \times \mathfrak{E}_c$.

Then: $\mathcal{K}(\mathfrak{E}_c)$ contains a dense subset homeomorphic to $\mathbb{Q} \times \mathfrak{E}_c$.

Hyperspaces and $\mathbb{Q} \times \mathfrak{E}_c$

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Question ([HG and Zaragoza, 2022])

Does $\mathfrak{E}_c^\omega$ contain a dense subset homeomorphic to $\mathbb{Q} \times \mathfrak{E}_c$?

Summary: hyperspaces

	$X = \mathfrak{E}$	$X = \mathfrak{E}_c$
$\mathcal{F}_n(X)$	homeomorphic to $\mathfrak{E}$	homeomorphic to $\mathfrak{E}_c$
$\mathcal{F}(X)$	homeomorphic to $\mathfrak{E}$	homeomorphic to $\mathbb{Q} \times \mathfrak{E}_c$
$\mathcal{K}(X)$	AZD, cohesive, non-Borel	AZD, cohesive, Polish
	homogeneous?	homeomorphic to $\mathfrak{E}_c$ or $\mathfrak{E}_c^\omega$?

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σ-product

Fix $e \in \mathfrak{E}_c$.

$$\sigma\mathfrak{E}_c^\omega = \{x \in \mathfrak{E}_c^\omega \colon \{n \in \omega \colon x_n \neq e\} \text{ is finite}\}.$$

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Theorem ([Lipham, preprint])

$\sigma\mathfrak{E}_c^\omega \in \sigma\mathcal{E}$; thus, $\sigma\mathfrak{E}_c^\omega \approx \mathbb{Q} \times \mathfrak{E}_c$.

σ-product

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Question ([Lipham, preprint])

Is the "pseudo-boundary" $\{x \in \mathfrak{E}_c^\omega \colon \exists n \in \omega\,(x_n = e)\}$ homeomorphic to $\mathbb{Q} \times \mathfrak{E}_c$?

Counterexample

Theorem ([Lipham, preprint])

There exists a separable metrizable space $E = \bigcup\{E_n \colon n \in \omega\}$ such that for each $n \in \omega$,

- *E_n is closed in E,*
- *$E_n \approx \mathfrak{E}_c$ and*
- *E_n is nowhere dense in E_{n+1},*

but $E \not\approx \mathbb{Q} \times \mathfrak{E}_c$.

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but $E \not\approx \mathbb{Q} \times \mathfrak{E}_c$.

Question ([HG and Zaragoza, 2022])

Let $X \subset \mathfrak{E}_c$ be a countable union of nowhere dense C-sets. If X is cohesive and dense, is $X \approx \mathbb{Q} \times \mathfrak{E}_c$?

Factors

A space X is a **factor** of a space Y is there is Z such that $X \times Z \approx Y$.

Examples:

- X is a factor of 2^ω iff X is compact, metrizable and zero-dimensional.

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- [Dijkstra, 2011] X is a factor of $\mathfrak{E}_c^\omega$ if and only if X is an AZD Polish space,

Factors

A space X is a **factor** of a space Y is there is Z such that $X \times Z \approx Y$.

Examples:

- X is a factor of 2^ω iff X is compact, metrizable and zero-dimensional.
- [Dijkstra and van Mill, 2009] X is a factor of $\mathfrak{E}_c$ if and only if X can be embedded as a closed set of $\mathfrak{E}_c$,
- [Dijkstra, 2011] X is a factor of $\mathfrak{E}_c^\omega$ if and only if X is an AZD Polish space,
- [Dijkstra and van Mill, 2010] X is a factor of $\mathfrak{E}$ if and only if X can be embedded as a closed set of $\mathfrak{E}$.

Factors of $\mathbb{Q} \times \mathfrak{E}_c$

$$\begin{array}{ccccc} \text{Factor of} & \Rightarrow & \text{Factor of} & \Rightarrow & \text{Factor of} \\ \mathfrak{E}_c & \not\Leftarrow & \mathbb{Q} \times \mathfrak{E}_c & \not\Leftarrow & \mathfrak{E} \end{array}$$

Factors of $\mathbb{Q} \times \mathfrak{E}_c$

Factor of $\mathfrak{E}_c$	$\Rightarrow$ / $\not\Leftarrow$	Factor of $\mathbb{Q} \times \mathfrak{E}_c$	$\Rightarrow$ / $\not\Leftarrow$	Factor of $\mathfrak{E}$

Theorem ([HG and Zaragoza, 2022])

A space E is a $(\mathbb{Q} \times \mathfrak{E}_c)$-factor if and only if there are a topology $\mathcal{W}$ on E witnessing that E is almost zero-dimensional, a collection of $\mathcal{W}$-closed non-empty subsets $\{E_n : n \in \omega\}$ and a basis of neighborhoods β such that

(i) $E = \bigcup\{E_n \colon n \in \omega\}$,

(ii) *for every* $n \in \omega$, $E_n \subset E_{n+1}$, *and*

(iii) *for every* $U \in \beta$ *and* $n \in \omega$, $U \cap E_n$ *is compact in* $\mathcal{W}$.

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Question ([HG and Zaragoza, 2022])

Is there a nicer characterization of $(\mathbb{Q} \times \mathfrak{E}_c)$-factors?

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Introduction to Erdős spaces

Hyperspaces

Characterization of $\mathbb{Q} \times \mathfrak{E}_c$

Questions

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Question: Knaster-Reichbach theorem analogue?

Theorem (Knaster and Reichbach, 1953)

If $A, B \subset 2^\omega$ are both closed and nowhere dense, then any homeomorphism $h\colon A \to B$ can be extended to a homeomorphism $H\colon 2^\omega \to 2^\omega$, $H \restriction A = h$.

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If $\tau \geq \omega$ and $A, B \subset 2^\tau$ are both closed and **negligible**, *then any homeomorphism $h\colon A \to B$ can be extended to a homeomorphism $H\colon 2^\tau \to 2^\tau$, $H \restriction A = h$.*

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Question

Let $A, B \subset \mathfrak{E}_c$ be closed, C-**sets** *and nowhere dense. If $h\colon A \to B$ is a homeomorphism, does there exist an extension $H\colon \mathfrak{E}_c \to \mathfrak{E}_c$, $H \restriction A = h$?*

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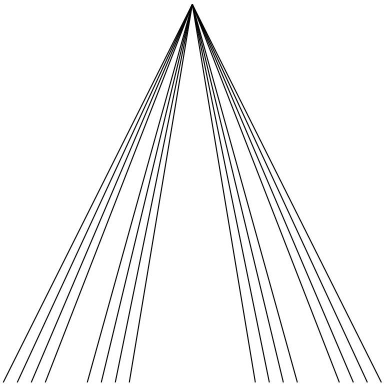
Smooth fans

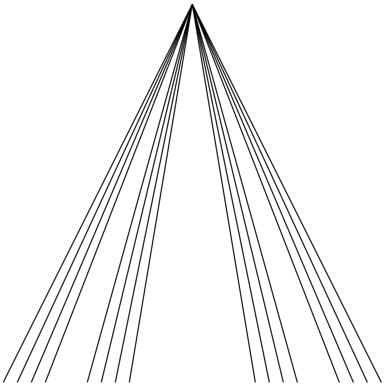


A **smooth fan** is any subcontinuum of the Cantor fan that is not a point or an arc.

The **Lelek fan** is **the*** smooth fan whose set of endpoints is dense in the Lelek fan.

*(Charatonik, 1989)

Smooth fans



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Theorem (Kawamura, Oversteegen and Tymchatyn, 1996)

The set of endpoints of the Lelek fan is homeomorphic to $\mathfrak{E}_c$.

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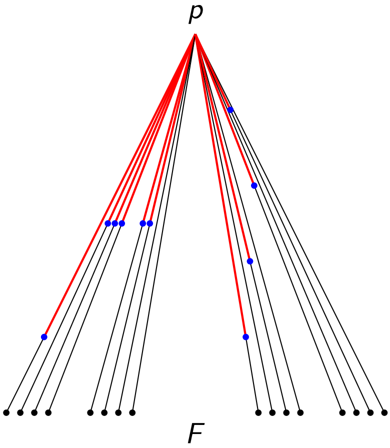
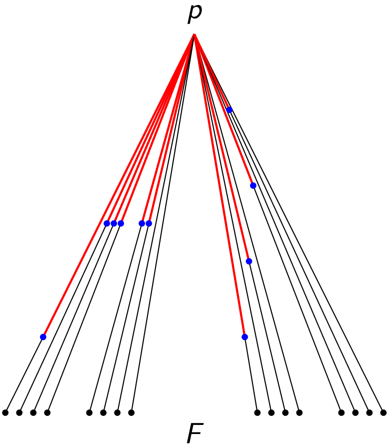
Smooth fans and AZD spaces



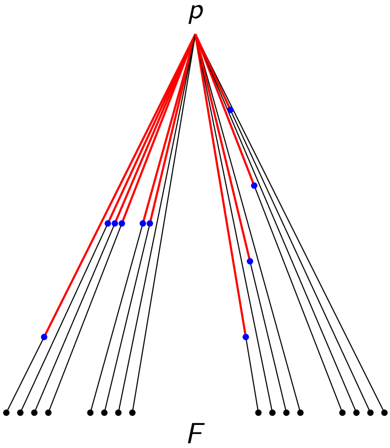
Let X be a smooth fan (embedded in the Cantor fan) and $E(X)$ its set of endpoints.

Smooth fans and AZD spaces



Let X be a smooth fan (embedded in the Cantor fan) and $E(X)$ its set of endpoints. Let p be the vertex of the Cantor fan, F the base at height 0.

Smooth fans and AZD spaces



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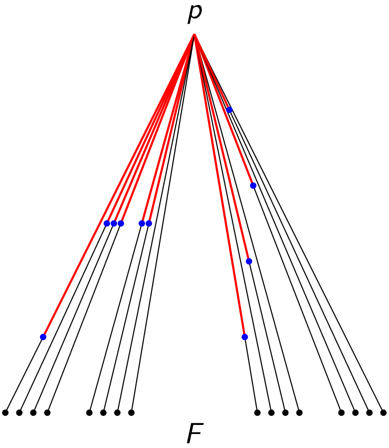
For every $e \in F$, let $\pi(e) \in F$ be such that p, e and $\pi(e)$ are colinear.

Smooth fans and AZD spaces



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For every $e \in F$, let $\pi(e) \in F$ be such that p, e and $\pi(e)$ are colinear. Then

$$\{\pi^{\leftarrow}(U) \colon U \text{ clopen in } F\}$$

is a witness topology for $E(X)$.

Smooth fans vs $\mathfrak{E}_c$

Theorem (folklore?; explict in [HG and Hoehn, preprint])

The set of endpoints of any smooth fan can be embedded as a closed subset of $\mathfrak{E}_c$.

Question ([HG and Hoehn, preprint])

Let E be a closed subset of $\mathfrak{E}_c$. Is there a smooth fan X such that $E(X) \approx E$?

Open questions

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i. Is $\mathcal{K}(\mathfrak{E}_c)$ homeomorphic to $\mathfrak{E}_c$ or $\mathfrak{E}_c^\omega$?

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vi. Find a "nice" characterization of factors of $\mathbb{Q} \times \mathfrak{E}_c$ that does not mention witness topologies, if possible.

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Problem

Study AZD spaces beyond the metric setting, for example, $\mathfrak{E}_c^{\omega_1}$.

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References

- Hernández-Gutiérrez, Rodrigo and Zaragoza, Alfredo; "A characterization of the product of the rational numbers and complete Erdős space." Canad. Math. Bull. (2022) 1-16. DOI: 10.4153/S0008439522000091
- Hernández-Gutiérrez, Rodrigo and Hoehn, Logan; "Smooth fans that are endpoint rigid." Preprint, .
- Lipham, David S.; "A characterization of Erdős space factors." Isr. J. Math. 246, No. 1 (2021), 395-402. DOI: 10.1007/s11856-021-2251-9
- Lipham, David S.; "The σ-product and CAP of complete Erdős space." Preprint, arXiv:2112.10172.
- A. Zaragoza, "Symmetric products of Erdős space and complete Erdős space." Topology Appl. 284 (2020), 107355, 10 pp. DOI: 10.1016/j.topol.2020.107355
- A. Zaragoza,"The Vietoris hyperspace of finite sets of Erdős space." Topology Appl. 310 (2021), 107943, 6 pp. DOI: 10.1016/j.topol.2021.107943

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