Some variations of the Banach-Mazur game

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Supported by FAPESP

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At the end, Bob is declared the winner if $\bigcap_{n\in\omega} B_n \neq \emptyset$.

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Theorem
If Bob *has a winning strategy on* BM(X), *then* X *is productively Baire (i.e.* $X \times Y$ *is Baire for every Baire* Y*).*

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Theorem
If BOB *has a winning strategy on* $\mathsf{BM}(X)$*, then* X *is productively Baire (i.e.* $X \times Y$ *is Baire for every Baire* Y*).*

Theorem
Let X *be a metrizable space with no isolated points. If* BOB *has a winning strategy on* $\mathsf{BM}(X)$*, then* X *contains a Cantor set.*

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It is well know that a Bernstein set is productively Baire - therefore it is not true that every productively Baire implies that Bob has a winning strategy in the Banach-Mazur game.

The implications

X is Baire $\Leftrightarrow$ $\textsc{Alice} \not\uparrow BM(X) \Leftarrow \textsc{Bob} \uparrow BM(X) \Rightarrow X$ is productively Baire

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Bob plays $B_0^0, ..., B_j^0 \subset A_0$, non-empty open sets;

In general, if $B_0^n, ..., B_k^n$ are the open sets played by Bob in the previous inning, then Alice plays

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Then, for each open set played by Alice, Bob plays finitely many non-empty open subsets.

For each inning n, let B_n be the union of all open sets played by Bob in that inning. At the end, Bob is declared the winner if $\bigcap_{n\in\omega} B_n \neq \emptyset$.

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Theorem
X is a Baire space if, and only if, Alice does not have a winning strategy for the $\mathrm{BM}_{\mathrm{fin}}(X)$ game.

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Playing $\mathrm{BM_{fin}}(\mathbb{R})$, Bob can assure that there is a Cantor set C inside of $\bigcap_{n\in\omega} B_n$.

Therefore, Bob has a winning strategy in the $\mathrm{BM_{fin}}(X)$ where X is a Bernstein set - since $C \cap X \neq \emptyset$.

But we still have some versions of Oxtoby's results:

Theorem
X is a Baire space if, and only if, Alice *does not have a winning strategy for the* $\mathrm{BM_{fin}}(X)$ *game.*

Theorem
If Bob *has a winning strategy on the* $\mathrm{BM_{fin}}(X)$ *game, then X is productively Baire.*

Is it a characterization?

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Unfortunately, the answer is no.

Making it even easier for Bob

Consider another variation for $\mathsf{BM}(X)$: exactly the same as the $\mathsf{BM}_{\mathrm{fin}}(X)$, but this time instead of being allowed to pick finitely many open sets each inning, now Bob can pick countable many open sets (all the other rules remain the same). This will be denoted by $\mathsf{BM}_{\omega}(X)$.

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But under CH (?) there is a subspace X of the real line that is Baire - therefore Bob has a winning strategy for the $\mathsf{BM}_\omega(X)$ game, such that Bob does not have a winning strategy for the $\mathsf{BM}_{\mathrm{fin}}(X)$.

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Theorem
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While, for BOB we actually don't know.

Question
Is it true that if BOB *has a winning strategy for* $\mathsf{BM}_\omega(X)$, *X is productively Baire?*

A particular case

It is know that every Baire space with a locally countable π-base (for instance, second countable spaces) is productively Baire.

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Theorem
Let X be a space with a locally countable π-base. If X is a Baire space, then Bob *has a winning strategy for the* $\mathsf{BM}_\omega(X)$ *game.*

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It is know that every Baire space with a locally countable π-base (for instance, second countable spaces) is productively Baire.

For this kind of space we can prove:

Theorem
Let X be a space with a locally countable π-base. If X is a Baire space, then Bob *has a winning strategy for the* $\mathsf{BM}_\omega(X)$ *game.*

Corollary
For spaces with a locally countable π-base, the game BM_ω *is determined.*

Bibliography

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