Some results and problems on Countable Dense Homogeneous spaces

Jan van Mill

University of Amsterdam
TU Delft

Twelfth Symposium on General Topology
and its Relations to Modern Analysis and Algebra
July 25-29, 2016, Prague

TOPOSYM 2016

1961 1966 1971 **1976** 1981 1986 1991 1996 2001 2006 2011 2016 stat

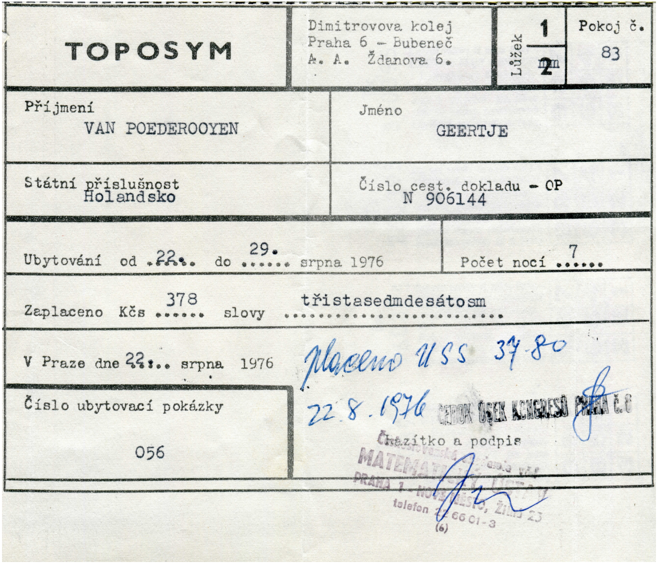
Fourth Symposium on General Topology

and its Relations to Modern Analysis and Algebra

Prague 1976

Jan van Mill, Jan van Wouwe and Geertje van Mill
1976

Hotel in 1976



TOPOSYM

Dimitrova kolej
Praha 6 – Bubeneč
A. A. Ždanova 6.

Lůžek 1 / 2

Pokoj č. 83

Příjmení VAN POEDEROOYEN

Jméno GEERTJE

Státní příslušnost Holandsko

Číslo cest. dokladu – OP N 906144

Ubytování od 22. do 29. srpna 1976

Počet nocí 7

Zaplaceno Kčs 378 slovy třistasedmdesátosm

V Praze dne 22. srpna 1976

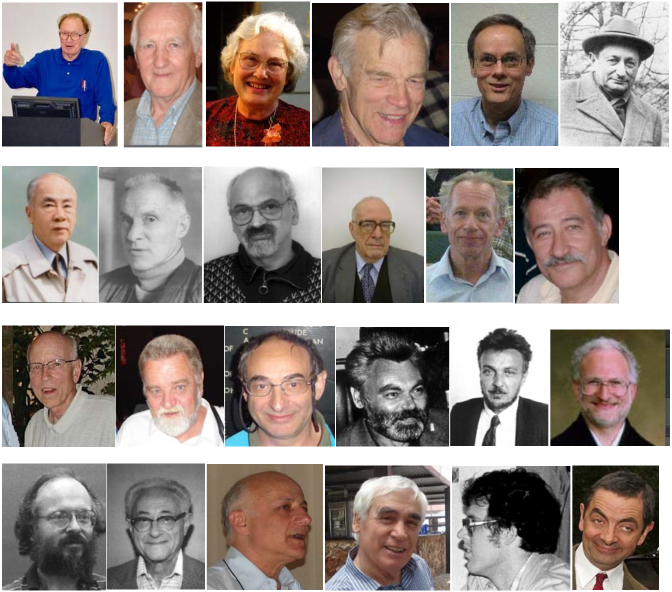
placeno USS 37.80
22.8.1976

ČEDOK ÚSEK KONGRESŮ PRAHA č. 6

Razítko a podpis

Číslo ubytovací pokázky 056

Documents



THE CLASS OF 1976

Say hello to all my friends in Prague! Tell them Brexit was not my idea!!!

- In the first part of the lecture, all spaces are *separable and metrizable*.

- In the first part of the lecture, all spaces are *separable and metrizable*.

Definition

A space X is *Countable Dense Homogeneous* (abbreviated: CDH) if given any two countable dense subsets D and E of X there is a homeomorphism $f\colon X \to X$ such that $f(D) = E$.

- In the first part of the lecture, all spaces are *separable and metrizable*.

Definition

A space X is *Countable Dense Homogeneous* (abbreviated: CDH) if given any two countable dense subsets D and E of X there is a homeomorphism $f\colon X \to X$ such that $f(D) = E$.

- There are many CDH-spaces: Cantor set, manifolds, Hilbert cube, etc. etc.

- In the first part of the lecture, all spaces are *separable and metrizable*.

Definition

A space X is *Countable Dense Homogeneous* (abbreviated: CDH) if given any two countable dense subsets D and E of X there is a homeomorphism $f\colon X \to X$ such that $f(D) = E$.

- There are many CDH-spaces: Cantor set, manifolds, Hilbert cube, etc. etc.
- 'Nice' spaces tend to be CDH.

- In the first part of the lecture, all spaces are *separable and metrizable*.

Definition

A space X is *Countable Dense Homogeneous* (abbreviated: CDH) if given any two countable dense subsets D and E of X there is a homeomorphism $f\colon X \to X$ such that $f(D) = E$.

- There are many CDH-spaces: Cantor set, manifolds, Hilbert cube, etc. etc.
- 'Nice' spaces tend to be CDH.
- Bennett proved in 1972 that connected (first-countable) CDH-spaces are homogeneous.

- Actually, connected CDH-spaces X are *n-homogeneous* for every n. That is, for all finite subsets $A, B \subseteq X$ such that $|A| = |B|$ there is a homeomorphism $f\colon X \to X$ such that $f(A) = B$ (vM, 2013).

- Actually, connected CDH-spaces X are *n-homogeneous* for every n. That is, for all finite subsets $A, B \subseteq X$ such that $|A| = |B|$ there is a homeomorphism $f\colon X \to X$ such that $f(A) = B$ (vM, 2013).
- Hence for connected spaces, CDH-ness can be thought of as a very strong form of homogeneity.

- Actually, connected CDH-spaces X are *n-homogeneous* for every n. That is, for all finite subsets $A, B \subseteq X$ such that $|A| = |B|$ there is a homeomorphism $f\colon X \to X$ such that $f(A) = B$ (vM, 2013).
- Hence for connected spaces, CDH-ness can be thought of as a very strong form of homogeneity.
- After 1972, the interest in CDH-spaces was kept alive mainly by Fitzpatrick.

Question (Fitzpatrick and Zhou, 1990)

Is every connected Polish CDH-space locally connected?

Question (Fitzpatrick and Zhou, 1990)

Is every connected Polish CDH-space locally connected?

- A Polish space is one that is (separable and) completely metrizable.

Question (Fitzpatrick and Zhou, 1990)

Is every connected Polish CDH-space locally connected?

- A Polish space is one that is (separable and) completely metrizable.
- Yes, for locally compact spaces (Fitzpatrick, 1972).

Question (Fitzpatrick and Zhou, 1990)

Is every connected Polish CDH-space locally connected?

- A Polish space is one that is (separable and) completely metrizable.
- Yes, for locally compact spaces (Fitzpatrick, 1972).

Theorem (vM, 2015)

Let X be a non-meager connected CDH*-space and assume that for some point x in X we have that for every open neighborhood W of x, the quasi-component of x in W is nontrivial. Then X is locally connected.*

- The quasi-component of x in X is the intersection of all open-and-closed subsets of X that contain x.

- The quasi-component of x in X is the intersection of all open-and-closed subsets of X that contain x.
- Hence the quasi-component of x contains the component of x.

- The quasi-component of x in X is the intersection of all open-and-closed subsets of X that contain x.
- Hence the quasi-component of x contains the component of x.
- The condition of the theorem says that some x in X has the following property: for every open neighborhood W of x there is a point $y \in W \setminus \{x\}$ so that x and y cannot be separated by (relative) clopen subsets of W.

- The quasi-component of x in X is the intersection of all open-and-closed subsets of X that contain x.
- Hence the quasi-component of x contains the component of x.
- The condition of the theorem says that some x in X has the following property: for every open neighborhood W of x there is a point $y \in W \setminus \{x\}$ so that x and y cannot be separated by (relative) clopen subsets of W.
- A counterexample to the Fitzpatrick-Zhou question (if it exists) must therefore be terrible: it is similar to a homogeneous version of the one-point connectification of complete *complete Erdős space*.

- The quasi-component of x in X is the intersection of all open-and-closed subsets of X that contain x.
- Hence the quasi-component of x contains the component of x.
- The condition of the theorem says that some x in X has the following property: for every open neighborhood W of x there is a point $y \in W \setminus \{x\}$ so that x and y cannot be separated by (relative) clopen subsets of W.
- A counterexample to the Fitzpatrick-Zhou question (if it exists) must therefore be terrible: it is similar to a homogeneous version of the one-point connectification of complete *complete Erdős space*.
- Complete Erdős space is the set of all vectors $x = (x_n)_n$ in Hilbert space ℓ^2 such that x_n is irrational for every n.

- The quasi-component of x in X is the intersection of all open-and-closed subsets of X that contain x.
- Hence the quasi-component of x contains the component of x.
- The condition of the theorem says that some x in X has the following property: for every open neighborhood W of x there is a point $y \in W \setminus \{x\}$ so that x and y cannot be separated by (relative) clopen subsets of W.
- A counterexample to the Fitzpatrick-Zhou question (if it exists) must therefore be terrible: it is similar to a homogeneous version of the one-point connectification of complete *complete Erdős space*.
- Complete Erdős space is the set of all vectors $x = (x_n)_n$ in Hilbert space ℓ^2 such that x_n is irrational for every n.
- It is totally disconnected (any two points can be separated by clopen sets) but 1-dimensional (Erdős, 1940).

- All of it nonempty clopen subsets have unbounded norm, and hence it can be made connected by the adjunction of a single point.

- All of it nonempty clopen subsets have unbounded norm, and hence it can be made connected by the adjunction of a single point.
- But the resulting space is not homogeneous.

- All of it nonempty clopen subsets have unbounded norm, and hence it can be made connected by the adjunction of a single point.
- But the resulting space is not homogeneous.
- The Erdős space is a very famous example in topology.

Question (Fitzpatrick and Zhou, 1990)

Does there exist a CDH-space that is not completely metrizable?

Question (Fitzpatrick and Zhou, 1990)

Does there exist a CDH-space that is not completely metrizable?

Theorem (Hrušák and Zamora Avilés, 2005)

Borel CDH-*spaces are Polish.*

Theorem (Farah, Hrušák and Martínez Ranero, 2005)

There is an absolute example of a CDH-*subspace of* $\mathbb{R}$ *of cardinality* $\aleph_1$.

Question (Fitzpatrick and Zhou, 1990)

Does there exist a CDH-space that is not completely metrizable?

Theorem (Hrušák and Zamora Avilés, 2005)

Borel CDH-*spaces are Polish.*

Theorem (Farah, Hrušák and Martínez Ranero, 2005)

There is an absolute example of a CDH-*subspace of* $\mathbb{R}$ *of cardinality* $\aleph_1$.

- A space X is called a λ-set if all of its countable subsets are G_δ.

Question (Fitzpatrick and Zhou, 1990)

Does there exist a CDH-space that is not completely metrizable?

Theorem (Hrušák and Zamora Avilés, 2005)

Borel CDH-*spaces are Polish.*

Theorem (Farah, Hrušák and Martínez Ranero, 2005)

There is an absolute example of a CDH-*subspace of* $\mathbb{R}$ *of cardinality* $\aleph_1$.

- A space X is called a λ-set if all of its countable subsets are G_δ.
- A crowded λ-set is meager (we will prove this in a moment).

Question (Fitzpatrick and Zhou, 1990)

Does there exist a CDH-space that is not completely metrizable?

Theorem (Hrušák and Zamora Avilés, 2005)

Borel CDH-*spaces are Polish.*

Theorem (Farah, Hrušák and Martínez Ranero, 2005)

There is an absolute example of a CDH-*subspace of* $\mathbb{R}$ *of cardinality* $\aleph_1$.

- A space X is called a λ-set if all of its countable subsets are G_δ.
- A crowded λ-set is meager (we will prove this in a moment).
- The space in the last theorem is a λ-set, hence is meager and so is not Polish.

Theorem

1. *There is a λ-set of size ω_1 (Lusin, 1921).*
2. *Every crowded λ-set is meager.*
3. *Every meager* CDH-*space is a λ-set (Fitzpatrick and Zhou, 1992).*

Theorem

1. *There is a λ-set of size ω_1 (Lusin, 1921).*
2. *Every crowded λ-set is meager.*
3. *Every meager* CDH-*space is a λ-set (Fitzpatrick and Zhou, 1992).*

Proof.

For (1), consider the quasi-order $\leq^*$ on ω^ω defined by

$$f \leq^* g \Leftrightarrow (\exists N < \omega)(\forall\, n \geq N)(f(n) \leq g(n)).$$

It is easy to construct a sequence $\{f_\alpha : \alpha < \omega_1\}$ of elements of ω^ω such that $f_\alpha <^* f_\beta$ for all $\alpha < \beta < \omega_1$. Then X is a λ-set in the subspace topology it inherits from ω^ω (with the standard Tychonoff product topology). □

Proof.

For (2), let X be a λ-set, and consider any countable dense subset D of X. Then D is G_δ, hence $X \setminus D$ is F_σ. All closed sets involved are nowhere dense.
For (3), let $\{B_n : n < \omega\}$ be a countable base for X consisting of nonempty sets. In addition, write X as $\bigcup_{n<\omega} F_n$, where each F_n is closed and nowhere dense. Pick a point $x_n \in B_n \setminus \bigcup_{i \le n} F_i$ for every n. Put $D = \{x_n : n < \omega\}$. Then $D \cap F_n$ is finite, hence $F_n \setminus D$ is F_σ, for every n. This shows that $X \setminus D$ is F_σ, hence D is G_δ. The rest follows from CDH-ness. □

Theorem (Hernández-Gutiérrez, Hrušák and vM, 2014)

For every uncountable cardinal $\kappa \leq \mathfrak{c}$, the following statements are equivalent:

1. *There is a meager* CDH*-space of size κ,*
2. *There is a λ-set of size κ.*

Theorem (Hernández-Gutiérrez, Hrušák and vM, 2014)

For every uncountable cardinal $\kappa \leq \mathfrak{c}$, the following statements are equivalent:

1. *There is a meager* CDH*-space of size κ,*
2. *There is a λ-set of size κ.*

- By an old result of Rothberger from 1939, this gives us:

Corollary

For every cardinal κ such that $\omega_1 \leq \kappa \leq \mathfrak{b}$ there exists a meager CDH*-space of size κ.*

Theorem (Hernández-Gutiérrez, Hrušák and vM, 2014)

For every uncountable cardinal $\kappa \leq \mathfrak{c}$, the following statements are equivalent:

1. *There is a meager* CDH*-space of size κ,*
2. *There is a λ-set of size κ.*

- By an old result of Rothberger from 1939, this gives us:

Corollary

For every cardinal κ such that $\omega_1 \leq \kappa \leq \mathfrak{b}$ there exists a meager CDH*-space of size κ.*

- Here $\mathfrak{b} = \min\{|B| : |B|$ is an unbounded subset of $\omega^\omega\}$. (With respect to the standard quasi-order that we defined above.)

- This motivates the question whether there is (in ZFC) a CDH-space of *any* cardinality below $\mathfrak{c}$.

- This motivates the question whether there is (in ZFC) a CDH-space of *any* cardinality below $\mathfrak{c}$.

Theorem (Hernández-Gutiérrez, Hrušák and vM, 2014)

It is consistent with ZFC *that the continuum is arbitrarily large and every* CDH*-space has size either* ω_1 *or* $\mathfrak{c}$*, and moreover*

1. *all* CDH*-spaces of size* ω_1 *are* λ*-sets, and*
2. *all* CDH*-spaces of size* $\mathfrak{c}$ *are non-meager.*

- This motivates the question whether there is (in ZFC) a CDH-space of *any* cardinality below $\mathfrak{c}$.

Theorem (Hernández-Gutiérrez, Hrušák and vM, 2014)

It is consistent with ZFC *that the continuum is arbitrarily large and every* CDH*-space has size either* ω_1 *or* $\mathfrak{c}$*, and moreover*

1. *all* CDH*-spaces of size* ω_1 *are* λ*-sets, and*
2. *all* CDH*-spaces of size* $\mathfrak{c}$ *are non-meager.*

- As we saw, there are spaces answering the Fitzpatrick-Zhou question that are not Polish because they are meager. How about Baire spaces?

- This motivates the question whether there is (in ZFC) a CDH-space of *any* cardinality below $\mathfrak{c}$.

Theorem (Hernández-Gutiérrez, Hrušák and vM, 2014)

It is consistent with ZFC *that the continuum is arbitrarily large and every* CDH*-space has size either* ω_1 *or* $\mathfrak{c}$*, and moreover*

1. *all* CDH*-spaces of size* ω_1 *are* λ*-sets, and*
2. *all* CDH*-spaces of size* $\mathfrak{c}$ *are non-meager.*

- As we saw, there are spaces answering the Fitzpatrick-Zhou question that are not Polish because they are meager. How about Baire spaces?

Theorem (Hernández-Gutiérrez, Hrušák and vM, 2014)

There is a CDH*-subspace of* $\mathbb{R}$ *which is Baire but not Polish.*

Question

Is it consistent with ZFC to have a (separable metric) Baire CDH-space without isolated points of size less than $\mathfrak{c}$?

Question

Can a nontrivial meager CDH-space be connected?

Question

Can a nontrivial meager CDH-space be connected?

- A nontrivial connected space has size $\mathfrak{c}$. Hence a positive answer to this question implies the existence of a λ-set of size $\mathfrak{c}$.

Question

Can a nontrivial meager CDH-space be connected?

- A nontrivial connected space has size $\mathfrak{c}$. Hence a positive answer to this question implies the existence of a λ-set of size $\mathfrak{c}$.
- By Miller (1993), the existence of a λ-set of size $\mathfrak{c}$ it is independent of ZFC.

Question

Can a nontrivial meager CDH-space be connected?

- A nontrivial connected space has size $\mathfrak{c}$. Hence a positive answer to this question implies the existence of a λ-set of size $\mathfrak{c}$.
- By Miller (1993), the existence of a λ-set of size $\mathfrak{c}$ it is independent of ZFC.
- Hence ZFC alone cannot prove the existence of a nontrivial connected meager CDH-space.

Question

Can a nontrivial meager CDH-space be connected?

- A nontrivial connected space has size $\mathfrak{c}$. Hence a positive answer to this question implies the existence of a λ-set of size $\mathfrak{c}$.
- By Miller (1993), the existence of a λ-set of size $\mathfrak{c}$ it is independent of ZFC.
- Hence ZFC alone cannot prove the existence of a nontrivial connected meager CDH-space.

Theorem (Hrušák and vM, 2016)

The following are equivalent:

1. *There is a λ-set of size $\mathfrak{c}$, and*
2. *there is a connected λ-set.*

Theorem (Hrušák and vM, 2016)

The Continuum Hypothesis (abbreviated: CH*) implies that there is a nontrivial meager connected* CDH*-subspace of the Hilbert cube* Q.

Theorem (Hrušák and vM, 2016)

The Continuum Hypothesis (abbreviated: CH*) implies that there is a nontrivial meager connected* CDH*-subspace of the Hilbert cube* Q.

Corollary (Hrušák and vM, 2016)

The existence of a nontrivial connected meager CDH*-space is independent of* ZFC

Theorem (Hrušák and vM, 2016)

The Continuum Hypothesis (abbreviated: CH) implies that there is a nontrivial meager connected CDH-subspace of the Hilbert cube Q.

Corollary (Hrušák and vM, 2016)

The existence of a nontrivial connected meager CDH-space is independent of ZFC

- The Hilbert cube Q is $\prod_{n=1}^{\infty}[-1,1]_n$.

Theorem (Hrušák and vM, 2016)

The Continuum Hypothesis (abbreviated: CH*) implies that there is a nontrivial meager connected* CDH*-subspace of the Hilbert cube* Q*.*

Corollary (Hrušák and vM, 2016)

The existence of a nontrivial connected meager CDH*-space is independent of* ZFC

- The Hilbert cube Q is $\prod_{n=1}^{\infty}[-1,1]_n$.
- $B(Q)$ is the so-called *pseudo-boundary* of Q, i.e.,

$$B(Q) = \{x \in Q : (\exists\, n \in \mathbb{N})(|x_n| = 1)\}.$$

Theorem (Hrušák and vM, 2016)

The Continuum Hypothesis (abbreviated: CH*) implies that there is a nontrivial meager connected* CDH*-subspace of the Hilbert cube* Q.

Corollary (Hrušák and vM, 2016)

The existence of a nontrivial connected meager CDH*-space is independent of* ZFC

- The Hilbert cube Q is $\prod_{n=1}^{\infty}[-1,1]_n$.
- $B(Q)$ is the so-called *pseudo-boundary* of Q, i.e.,

$$B(Q) = \{x \in Q : (\exists n \in \mathbb{N})(|x_n| = 1)\}.$$

- The proof of the theorem uses the following results:

Lemma

Let A be a G_δ-subset of $[-1,1]$ such that $[-1,1] \setminus A \neq \emptyset$. Then there is a homeomorphism $f\colon Q \to Q$ such that $f(B(Q)) = B(Q) \setminus A^\infty$.

Lemma

Let A be a G_δ-subset of $[-1,1]$ such that $[-1,1] \setminus A \neq \emptyset$. Then there is a homeomorphism $f\colon Q \to Q$ such that $f(B(Q)) = B(Q) \setminus A^\infty$.

- A subset B of Q for which there exists a homeomorphism $f\colon Q \to Q$ such that $f(B) = B(Q)$ is called a *capset*.

Lemma

Let A be a G_δ-subset of $[-1, 1]$ such that $[-1, 1] \setminus A \neq \emptyset$. Then there is a homeomorphism $f : Q \to Q$ such that $f(B(Q)) = B(Q) \setminus A^\infty$.

- A subset B of Q for which there exists a homeomorphism $f : Q \to Q$ such that $f(B) = B(Q)$ is called a *capset*.

Lemma

Let M and N be capsets in Q. In addition, let D^0 be a countable dense subset of $Q \setminus M$ containing the dense subset E^0 such that $F^0 = D^0 \setminus E^0$ is dense as well. Moreover, let D^1 be a countable dense subset of $Q \setminus N$ containing the dense subset E^1 such that $F^1 = D^1 \setminus E^1$ is dense as well. Then there is a homeomorphism h of Q such that $h(M) = N$, $h(E^0) = E^1$ and $h(F^0) = F^1$.

- So assume CH, and write $[-1,1]$ as $\bigcup_{\alpha<\omega_1} A_\alpha$, so that $A_0=\emptyset$, each A_α is a G_δ-subset of $[-1,1]$, $A_\alpha \subseteq A_\beta$ if $\alpha<\beta$, and $[-1,1]\setminus A_\alpha \neq \emptyset$.

- So assume CH, and write $[-1,1]$ as $\bigcup_{\alpha<\omega_1} A_\alpha$, so that $A_0=\emptyset$, each A_α is a G_δ-subset of $[-1,1]$, $A_\alpha \subseteq A_\beta$ if $\alpha<\beta$, and $[-1,1]\setminus A_\alpha \neq \emptyset$.
- Enumerate all closed subsets of Q that separate Q by $\{K_\alpha : \alpha<\omega_1\}$, and enumerate all pairs of countable dense subsets of Q by $\{(E_\alpha, F_\alpha) : \alpha<\omega_1\}$ such that each pair is listed ω_1-many times.

- So assume CH, and write $[-1,1]$ as $\bigcup_{\alpha<\omega_1} A_\alpha$, so that $A_0=\emptyset$, each A_α is a G_δ-subset of $[-1,1]$, $A_\alpha \subseteq A_\beta$ if $\alpha<\beta$, and $[-1,1]\setminus A_\alpha \neq \emptyset$.
- Enumerate all closed subsets of Q that separate Q by $\{K_\alpha : \alpha<\omega_1\}$, and enumerate all pairs of countable dense subsets of Q by $\{(E_\alpha, F_\alpha) : \alpha<\omega_1\}$ such that each pair is listed ω_1-many times.
- We shall recursively construct a decreasing sequence $\{B_\alpha : \alpha<\omega_1\}$ of capsets and an increasing sequence $\{D_\alpha : \alpha<\omega_1\}$ of countable subsets of Q, together with an increasing sequence $\{H_\alpha : \alpha<\omega_1\}$ of countable subgroups of $\mathcal{H}(Q)$ so that (denoting $Q\setminus B_\alpha$ by s_α) for every $\alpha<\omega_1$:

1. D_α is a countable dense subset of s_α, and $D_\alpha \cap K_\alpha \neq \emptyset$,

1. D_α is a countable dense subset of s_α, and $D_\alpha \cap K_\alpha \neq \emptyset$,
2. there exists an ordinal $f(\alpha) < \omega_1$ such that $B(Q) \setminus A^\infty_{f(\alpha)} \subseteq B_\alpha$,

1. D_α is a countable dense subset of s_α, and $D_\alpha \cap K_\alpha \neq \emptyset$,
2. there exists an ordinal $f(\alpha) < \omega_1$ such that $B(Q) \setminus A^{\infty}_{f(\alpha)} \subseteq B_\alpha$,
3. D_α, s_α and B_α are invariant under H_α,

1. D_α is a countable dense subset of s_α, and $D_\alpha \cap K_\alpha \neq \emptyset$,
2. there exists an ordinal $f(\alpha) < \omega_1$ such that $B(Q) \setminus A^\infty_{f(\alpha)} \subseteq B_\alpha$,
3. D_α, s_α and B_α are invariant under H_α,
4. if $E_\alpha \cup F_\alpha \subseteq D_\alpha$, and $D_\alpha \setminus (E_\alpha \cup F_\alpha)$ is dense, then there exists an element h of H_α such that $h(E_\alpha) = F_\alpha$,

1. D_α is a countable dense subset of s_α, and $D_\alpha \cap K_\alpha \neq \emptyset$,
2. there exists an ordinal $f(\alpha) < \omega_1$ such that $B(Q) \setminus A^\infty_{f(\alpha)} \subseteq B_\alpha$,
3. D_α, s_α and B_α are invariant under H_α,
4. if $E_\alpha \cup F_\alpha \subseteq D_\alpha$, and $D_\alpha \setminus (E_\alpha \cup F_\alpha)$ is dense, then there exists an element h of H_α such that $h(E_\alpha) = F_\alpha$,
5. if $\gamma < \alpha$, $D_\alpha \setminus D_\gamma$ is a dense subset of Q contained in $s_\alpha \setminus s_\gamma$.

1. D_α is a countable dense subset of s_α, and $D_\alpha \cap K_\alpha \neq \emptyset$,
2. there exists an ordinal $f(\alpha) < \omega_1$ such that $B(Q) \setminus A^\infty_{f(\alpha)} \subseteq B_\alpha$,
3. D_α, s_α and B_α are invariant under H_α,
4. if $E_\alpha \cup F_\alpha \subseteq D_\alpha$, and $D_\alpha \setminus (E_\alpha \cup F_\alpha)$ is dense, then there exists an element h of H_α such that $h(E_\alpha) = F_\alpha$,
5. if $\gamma < \alpha$, $D_\alpha \setminus D_\gamma$ is a dense subset of Q contained in $s_\alpha \setminus s_\gamma$.
6. if $\gamma < \alpha$, then H_γ is a subgroup of H_α.

1. D_α is a countable dense subset of s_α, and $D_\alpha \cap K_\alpha \neq \emptyset$,
2. there exists an ordinal $f(\alpha) < \omega_1$ such that $B(Q) \setminus A^\infty_{f(\alpha)} \subseteq B_\alpha$,
3. D_α, s_α and B_α are invariant under H_α,
4. if $E_\alpha \cup F_\alpha \subseteq D_\alpha$, and $D_\alpha \setminus (E_\alpha \cup F_\alpha)$ is dense, then there exists an element h of H_α such that $h(E_\alpha) = F_\alpha$,
5. if $\gamma < \alpha$, $D_\alpha \setminus D_\gamma$ is a dense subset of Q contained in $s_\alpha \setminus s_\gamma$.
6. if $\gamma < \alpha$, then H_γ is a subgroup of H_α.

- Then $D = \bigcup_{\alpha < \omega_1} D_\alpha$ is the example we are looking for.

Question

Is there, assuming CH, a connected meager CDH-space in the plane?

Question

Is there, assuming CH, a connected meager CDH-space in the plane?

Question

Is it consistent with ZFC that there is a connected λ-set yet there is no connected meager CDH-space?

- A space X is called *Strongly Locally Homogeneous* (abbreviated: SLH) if it has an open base $\mathcal{B}$ such that for all $B \in \mathcal{B}$ and $x, y \in B$ there is a homeomorphism $f\colon X \to X$ such that $f(x) = y$ and $f(z) = z$ for every $z \notin B$.

- A space X is called *Strongly Locally Homogeneous* (abbreviated: SLH) if it has an open base $\mathcal{B}$ such that for all $B \in \mathcal{B}$ and $x, y \in B$ there is a homeomorphism $f\colon X \to X$ such that $f(x) = y$ and $f(z) = z$ for every $z \notin B$.
- (Bessaga and Pełczyński, 1970) Every Polish SLH-space is CDH.

- A space X is called *Strongly Locally Homogeneous* (abbreviated: SLH) if it has an open base $\mathcal{B}$ such that for all $B \in \mathcal{B}$ and $x, y \in B$ there is a homeomorphism $f\colon X \to X$ such that $f(x) = y$ and $f(z) = z$ for every $z \notin B$.
- (Bessaga and Pełczyński, 1970) Every Polish SLH-space is CDH.

Theorem (Kennedy, 1984)

A 2-homogeneous continuum X must be SLH, *provided that X admits a nontrivial homeomorphism that is the identity on some nonempty open set.*

- A space X is called *Strongly Locally Homogeneous* (abbreviated: SLH) if it has an open base $\mathcal{B}$ such that for all $B \in \mathcal{B}$ and $x, y \in B$ there is a homeomorphism $f\colon X \to X$ such that $f(x) = y$ and $f(z) = z$ for every $z \notin B$.
- (Bessaga and Pełczyński, 1970) Every Polish SLH-space is CDH.

Theorem (Kennedy, 1984)

A 2-homogeneous continuum X must be SLH, *provided that X admits a nontrivial homeomorphism that is the identity on some nonempty open set.*

- Observe that for compact spaces, SLH $\Rightarrow$ CDH (Bessaga and Pełczyński).

- A space X is called *Strongly Locally Homogeneous* (abbreviated: SLH) if it has an open base $\mathcal{B}$ such that for all $B \in \mathcal{B}$ and $x, y \in B$ there is a homeomorphism $f\colon X \to X$ such that $f(x) = y$ and $f(z) = z$ for every $z \notin B$.
- (Bessaga and Pełczyński, 1970) Every Polish SLH-space is CDH.

Theorem (Kennedy, 1984)

A 2-homogeneous continuum X must be SLH, *provided that X admits a nontrivial homeomorphism that is the identity on some nonempty open set.*

- Observe that for compact spaces, SLH $\Rightarrow$ CDH (Bessaga and Pełczyński).
- CDH and connected $\Rightarrow$ n-homogeneous for every n (vM).

- A space X is called *Strongly Locally Homogeneous* (abbreviated: SLH) if it has an open base $\mathcal{B}$ such that for all $B \in \mathcal{B}$ and $x, y \in B$ there is a homeomorphism $f\colon X \to X$ such that $f(x) = y$ and $f(z) = z$ for every $z \notin B$.
- (Bessaga and Pełczyński, 1970) Every Polish SLH-space is CDH.

Theorem (Kennedy, 1984)

A 2-homogeneous continuum X must be SLH, *provided that X admits a nontrivial homeomorphism that is the identity on some nonempty open set.*

- Observe that for compact spaces, SLH $\Rightarrow$ CDH (Bessaga and Pełczyński).
- CDH and connected $\Rightarrow$ n-homogeneous for every n (vM).
- Compact + 2-homogeneous + $\exists$ a special homeomorphism $\Rightarrow$ SLH (Kennedy).

- Hence for continua admitting such a homeomorphism we have: SLH $\Leftrightarrow$ 2-homogeneous $\Leftrightarrow$ CDH.

- Hence for continua admitting such a homeomorphism we have: SLH $\Leftrightarrow$ 2-homogeneous $\Leftrightarrow$ CDH.

Question

Does every 2-homogeneous continuum admit such a homeomorphism?

- Hence for continua admitting such a homeomorphism we have: SLH $\Leftrightarrow$ 2-homogeneous $\Leftrightarrow$ CDH.

Question

Does every 2-homogeneous continuum admit such a homeomorphism?

- Compactness is essential in this problem.

Theorem (vM, 2005)

There is a connected, Polish, CDH*-space* X *that is not* SLH*. In fact, a homeomorphism on* X *that is the identity on some nonempty open subset of* X *must be the identity on all of* X*.*

- Now we leave the separable metrizable world, from now on all spaces under discussion are *Tychonoff*.

- Now we leave the separable metrizable world, from now on all spaces under discussion are *Tychonoff*.
- Are there in ZFC compact CDH-spaces that are not metrizable?

- Now we leave the separable metrizable world, from now on all spaces under discussion are *Tychonoff*.
- Are there in ZFC compact CDH-spaces that are not metrizable?

Theorem (Steprans and Zhou, 1988)

2^κ *is* CDH *for every* $\kappa < \mathfrak{p}$*.*

- Now we leave the separable metrizable world, from now on all spaces under discussion are *Tychonoff*.
- Are there in ZFC compact CDH-spaces that are not metrizable?

Theorem (Steprans and Zhou, 1988)

2^κ *is* CDH *for every* $\kappa < \mathfrak{p}$.

- Here $\mathfrak{p} = \min\{|\mathcal{F}| : \mathcal{F}$ is a subfamily of $[\omega]^\omega$ with the sfip which has no infinite pseudo-intersection$\}$.

- Now we leave the separable metrizable world, from now on all spaces under discussion are *Tychonoff*.
- Are there in ZFC compact CDH-spaces that are not metrizable?

Theorem (Steprans and Zhou, 1988)

2^κ *is* CDH *for every* $\kappa < \mathfrak{p}$.

- Here $\mathfrak{p} = \min\{|\mathcal{F}| : \mathcal{F}$ is a subfamily of $[\omega]^\omega$ with the sfip which has no infinite pseudo-intersection$\}$.
- Under Martin's Axiom, abbreviated MA, $\mathfrak{p} = \mathfrak{c}$.

- Now we leave the separable metrizable world, from now on all spaces under discussion are *Tychonoff*.
- Are there in ZFC compact CDH-spaces that are not metrizable?

Theorem (Steprans and Zhou, 1988)

2^κ *is* CDH *for every* $\kappa < \mathfrak{p}$*.*

- Here $\mathfrak{p} = \min\{|\mathcal{F}| : \mathcal{F}$ is a subfamily of $[\omega]^\omega$ with the sfip which has no infinite pseudo-intersection$\}$.
- Under Martin's Axiom, abbreviated MA, $\mathfrak{p} = \mathfrak{c}$.

Corollary (Steprans and Zhou, 1988)

Under MA$+\neg$CH, 2^{ω_1} *is* CDH.

Theorem

1. *(Arhangel'skii and vM, 2013) Under* CH, *there is a compact* CDH*-space of uncountable weight. In fact, it is both hereditarily Lindelöf and hereditarily separable.*

Theorem

1. *(Arhangel'skii and vM, 2013) Under* CH, *there is a compact* CDH-*space of uncountable weight. In fact, it is both hereditarily Lindelöf and hereditarily separable.*
2. *(Arhangel'skii and vM, 2013) Under* $\mathfrak{c} < 2^{\omega_1}$, *every compact* CDH-*space is first-countable.*

Theorem

1. *(Arhangel'skii and vM, 2013) Under* CH, *there is a compact* CDH*-space of uncountable weight. In fact, it is both hereditarily Lindelöf and hereditarily separable.*
2. *(Arhangel'skii and vM, 2013) Under* $\mathfrak{c} < 2^{\omega_1}$, *every compact* CDH*-space is first-countable.*
3. *(Arhangel'skii and vM, 2013) The Alexandroff-Urysohn double is not* CDH.

Theorem

1. *(Arhangel'skii and vM, 2013) Under* CH, *there is a compact* CDH-*space of uncountable weight. In fact, it is both hereditarily Lindelöf and hereditarily separable.*
2. *(Arhangel'skii and vM, 2013) Under* $\mathfrak{c} < 2^{\omega_1}$, *every compact* CDH-*space is first-countable.*
3. *(Arhangel'skii and vM, 2013) The Alexandroff-Urysohn double is not* CDH.

Theorem (Hernández-Gutiérrez, 2013)

The Alexandroff-Urysohn double has $\mathfrak{c}$ *types of countable dense sets.*

Theorem (Hernández-Gutiérrez, Hrušák and vM, 2014)

The double arrow space over a saturated λ'-set Y is a compact CDH*-space of weight $|A|$.*

Theorem (Hernández-Gutiérrez, Hrušák and vM, 2014)

The double arrow space over a saturated λ'-set Y is a compact CDH*-space of weight* $|A|$.

Corollary

There exists a linearly ordered, compact, zero-dimensional CDH*-space of weight* ω_1.

Theorem (Hernández-Gutiérrez, Hrušák and vM, 2014)

The double arrow space over a saturated λ'-set Y is a compact CDH-*space of weight* $|A|$.

Corollary

There exists a linearly ordered, compact, zero-dimensional CDH-*space of weight* ω_1.

Question

Is there a compact CDH-space of weight $\mathfrak{c}$ in ZFC?

Theorem (Hernández-Gutiérrez, Hrušák and vM, 2014)

The double arrow space over a saturated λ'-set Y is a compact CDH*-space of weight* $|A|$.

Corollary

There exists a linearly ordered, compact, zero-dimensional CDH*-space of weight* ω_1.

Question

Is there a compact CDH-space of weight $\mathfrak{c}$ in ZFC?

Question

Is there a non-metrizable CDH-continuum?

THANK YOU FOR YOUR ATTENTION!