Random elements of large groups: Discrete case

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Toposym 2016

joint work with

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The random graph, $\mathcal{R} = \langle \mathbb{N}, E_{\mathcal{R}} \rangle$

Edges: for $n, m \in \mathbb{N}$ distinct let $\mathbb{P}((n,m) \in E_{\mathcal{R}}) = \frac{1}{2}$, independently.

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$\langle \mathbb{Q}, < \rangle$

If $X, Y \subset \mathbb{Q}$ are finite and $f : X \to Y$ is order preserving then f extends to an order preserving $\mathbb{Q} \to \mathbb{Q}$ map.

Every countable linearly ordered set can be order preservingly embedded to $\mathbb{Q}$.

Automorphism groups and genericity

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Definition. If $f, g \in Aut(\mathcal{A})$ we say that f and g are *conjugate*, if there exists an $h \in Aut(\mathcal{A})$ such that $h^{-1}fh = g$.

Note: if $f, g \in Aut(\mathcal{A})$ then

$$\langle \mathcal{A}, f \rangle \cong \langle \mathcal{A}, g \rangle \iff (\exists h \in Aut(\mathcal{A}))(h^{-1}fh = g).$$

Definition. An automorphism is called *generic* if its conjugacy class is co-meagre.

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- (Kuske, Truss) There exist generic elements in $Aut(\mathbb{Q})$ and $Aut(\mathcal{R})$.

Kechris, Rosendal: Characterisation of the existence of generic elements of closed subgroups of S_∞.

Measure

Definition. (Christensen) Let $(G, \cdot)$ be a Polish group and $B \subset G$ Borel. We say that B is *Haar null* if there exists a Borel probability measure μ on G such that for every $g, h \in G$ we have $\mu(gBh) = 0$. An arbitrary set S is called Haar null if $S \subset B$ for some Borel Haar null set B.

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Corollary. If A is compact biter then it is not Haar null.

Measure in S_∞

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Theorem. (Dougherty, Mycielski) All of these classes are Haar positive, in fact, compact biters.

Measure and countable structures

Definition. Let $\mathcal{A}$ be a structure, $a \in A$ and $X \subset A$. We say that $\mathcal{A}$ *has the nice algebraic closure property (NAC)* if for every finite $A \subset \mathcal{A}$ the $\{b : |\{f(b) : f \in Stab_p(A)\}| < \infty\}$ is finite.

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$\mathcal{R}, \mathbb{Q}$ has NAC, but this is not enough to characterize the positive conjugacy classes of $Aut(\mathcal{R})$, $Aut(\mathbb{Q})$.

Measure and $Aut(\mathbb{Q})$

$f \in Aut(\mathbb{Q})$ extends to a $\bar{f} \in Homeo^+(\mathbb{R})$.
Definition. A $+$ *orbital* ($-$ *orbital*) of f is a maximal interval $I \subset \mathbb{R}$ such that for every $x \in I$ we have $\bar{f}(x) > x$ ($\bar{f}(x) < x$).
Let $Fix(\bar{f}) = \{x \in \mathbb{R} : \bar{f}(x) = x\}$.

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Let $Fix(\bar{f}) = \{x \in \mathbb{R} : \bar{f}(x) = x\}$.
Proposition. $f, g \in Aut(\mathbb{Q})$ are conjugate if and only if there exists an order and rationality preserving isomorphism between $Fix(\bar{f})$ and $Fix(\bar{g})$ so that the corresponding orbitals have the same sign.

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Theorem. For almost every element of $Aut(\mathbb{Q})$

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Definition. Let $v \in R$ and $f \in Aut(\mathcal{R})$. Define $\beta_{f,v} : \mathbb{N}^+ \to \{0,1\}$ as

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Proposition. (Truss) Suppose that $f, g \in Aut(\mathcal{R})$ have only one infinite cycle and no finite ones. Then f and g are conjugate if and only if $\beta_{f,v} = \beta_{g,w}$ for some ($\iff$ for every) v, w.

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Splitting lemma. If $F \subset Aut(\mathcal{R})$ is finite set there exists a vertex v so that for every $f, g \in F$ distinct we have $f(v) \neq g(v)$.

Measure and $Aut(\mathcal{R})$

Theorem. (Christensen) If A is a conjugacy invariant Haar positive universally measurable set then $A^{-1}A$ contains a neighbourhood of the identity.
Corollary. (Truss) For every $f, g \in Aut(\mathcal{R})$ non-identity elements, g is the product of four conjugates of f.

Questions

1. How many Haar positive conjugacy classes are there?

2. Is the union of the Haar null conjugacy classes Haar null?

Examples

	$\bigcup$ of Haar null classes is Haar null		
	C	LC $\setminus$ C	NLC
0			
n			
$\aleph_0$			
$\mathfrak{c}$			
	$\bigcup$ of Haar null classes is not Haar null		
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	C	LC \ C	NLC
0	–	–	–
n	$\mathbb{Z}_n$	HNN	???
$\aleph_0$	???	$\mathbb{Z}$	S_∞
$\mathfrak{c}$	–	–	$Aut(\mathbb{Q})$; $Aut(\mathcal{R})$
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	C	LC \ C	NLC
0	2^ω	$\mathbb{Z} \times 2^\omega$	$\mathbb{Z}^\omega$
n	$\mathbb{Z}_n \times (\mathbb{Z}_2 \ltimes \mathbb{Z}_3^\omega)$	HNN $\times (\mathbb{Z}_2 \ltimes \mathbb{Z}_3^\omega)$	$\mathbb{Z}_n \times (\mathbb{Z}_2 \ltimes \mathbb{Q}_d^\omega)$
$\aleph_0$	???	$\mathbb{Z} \times (\mathbb{Z}_2 \ltimes \mathbb{Z}_3^\omega)$	$S_\infty \times (\mathbb{Z}_2 \ltimes \mathbb{Z}_3^\omega)$
$\mathfrak{c}$	–	–	$Aut(\mathbb{Q}) \times (\mathbb{Z}_2 \ltimes \mathbb{Z}_3^\omega)$

Open problems

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Question. Does there exist a Polish group such that it consistently has κ many Haar positive conjugacy classes with $\aleph_0 < \kappa < \mathfrak{c}$?
Problem. Formulate necessary and sufficient model theoretic conditions which characterise the measure theoretic behaviour of the conjugacy classes!

Thank you for your attention!