Productively (and non-productively) Menger spaces

Piotr Szewczak

Cardinal Stefan Wyszyński University, Poland, and Bar-Ilan University, Israel

joint work with Boaz Tsaban

Toposym 2016

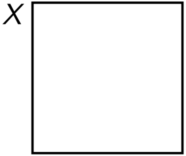
Supported by National Science Center Poland UMO-2014/12/T/ST1/00627

The Menger property

Menger's property: for every sequence of open covers $\mathcal{O}_1, \mathcal{O}_2, \ldots$ of X there are finite $\mathcal{F}_1 \subseteq \mathcal{O}_1, \mathcal{F}_2 \subseteq \mathcal{O}_2, \ldots$ such that $\mathcal{F}_1 \cup \mathcal{F}_2 \cup \ldots$ covers X

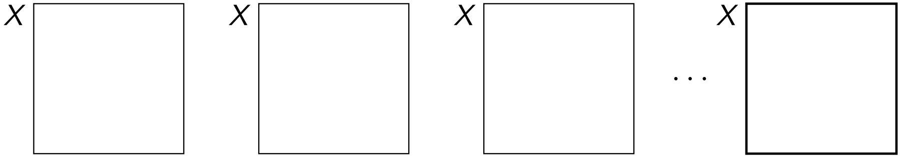
The Menger property

Menger's property: for every sequence of open covers $\mathcal{O}_1, \mathcal{O}_2, \ldots$ of X there are finite $\mathcal{F}_1 \subseteq \mathcal{O}_1, \mathcal{F}_2 \subseteq \mathcal{O}_2, \ldots$ such that $\mathcal{F}_1 \cup \mathcal{F}_2 \cup \ldots$ covers X



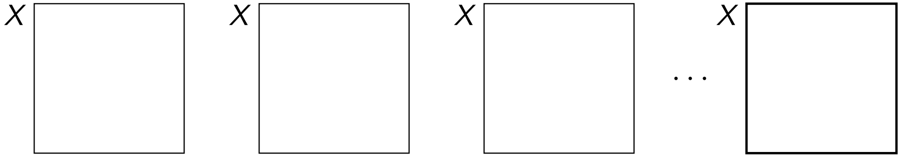
The Menger property

Menger's property: for every sequence of open covers $\mathcal{O}_1, \mathcal{O}_2, \ldots$ of X there are finite $\mathcal{F}_1 \subseteq \mathcal{O}_1, \mathcal{F}_2 \subseteq \mathcal{O}_2, \ldots$ such that $\mathcal{F}_1 \cup \mathcal{F}_2 \cup \ldots$ covers X



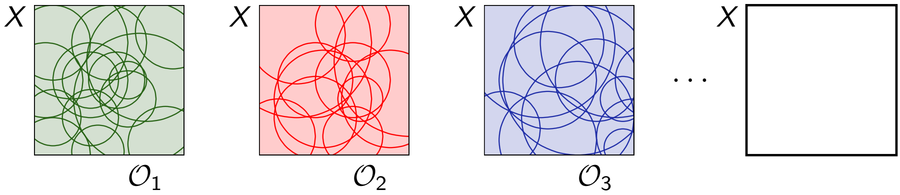
The Menger property

Menger's property: for every sequence of open covers $\mathcal{O}_1, \mathcal{O}_2, \ldots$ of X there are finite $\mathcal{F}_1 \subseteq \mathcal{O}_1, \mathcal{F}_2 \subseteq \mathcal{O}_2, \ldots$ such that $\mathcal{F}_1 \cup \mathcal{F}_2 \cup \ldots$ covers X



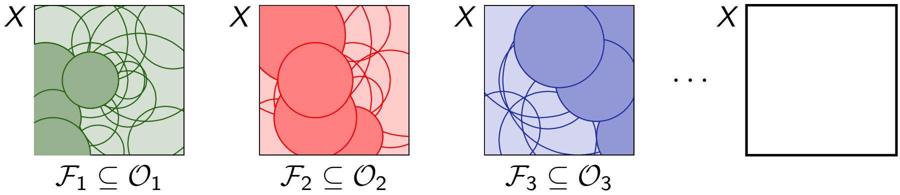
The Menger property

Menger's property: for every sequence of open covers $\mathcal{O}_1, \mathcal{O}_2, \ldots$ of X there are finite $\mathcal{F}_1 \subseteq \mathcal{O}_1, \mathcal{F}_2 \subseteq \mathcal{O}_2, \ldots$ such that $\mathcal{F}_1 \cup \mathcal{F}_2 \cup \ldots$ covers X



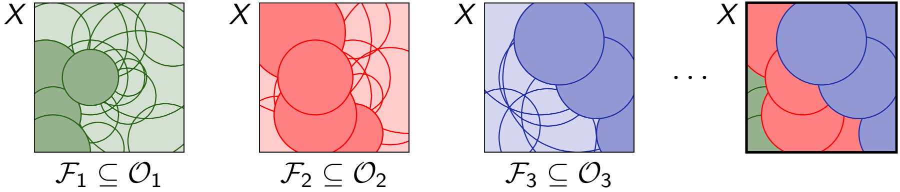
The Menger property

Menger's property: for every sequence of open covers $\mathcal{O}_1, \mathcal{O}_2, \ldots$ of X there are finite $\mathcal{F}_1 \subseteq \mathcal{O}_1, \mathcal{F}_2 \subseteq \mathcal{O}_2, \ldots$ such that $\mathcal{F}_1 \cup \mathcal{F}_2 \cup \ldots$ covers X



The Menger property

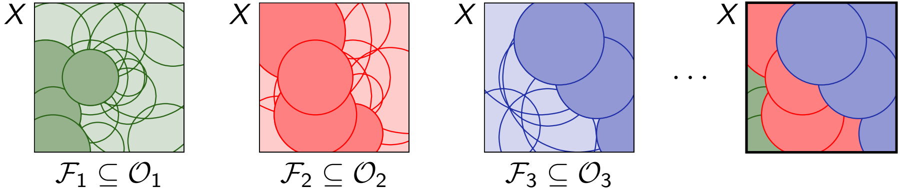
Menger's property: for every sequence of open covers $\mathcal{O}_1, \mathcal{O}_2, \ldots$ of X there are finite $\mathcal{F}_1 \subseteq \mathcal{O}_1, \mathcal{F}_2 \subseteq \mathcal{O}_2, \ldots$ such that $\mathcal{F}_1 \cup \mathcal{F}_2 \cup \ldots$ covers X



Menger $\Rightarrow$ Lindelöf

The Menger property

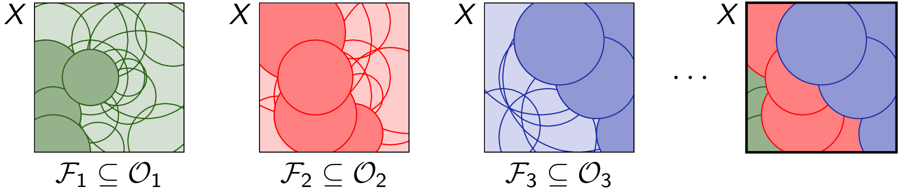
Menger's property: for every sequence of open covers $\mathcal{O}_1, \mathcal{O}_2, \ldots$ of X there are finite $\mathcal{F}_1 \subseteq \mathcal{O}_1, \mathcal{F}_2 \subseteq \mathcal{O}_2, \ldots$ such that $\mathcal{F}_1 \cup \mathcal{F}_2 \cup \ldots$ covers X



σ-compact $\Rightarrow$ Menger $\Rightarrow$ Lindelöf

The Menger property

Menger's property: for every sequence of open covers $\mathcal{O}_1, \mathcal{O}_2, \ldots$ of X there are finite $\mathcal{F}_1 \subseteq \mathcal{O}_1, \mathcal{F}_2 \subseteq \mathcal{O}_2, \ldots$ such that $\mathcal{F}_1 \cup \mathcal{F}_2 \cup \ldots$ covers X

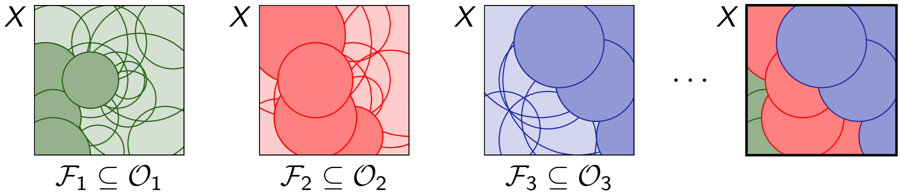


σ-compact $\Rightarrow$ Menger $\Rightarrow$ Lindelöf

Aurichi: Every Menger space is D

The Menger property

Menger's property: for every sequence of open covers $\mathcal{O}_1, \mathcal{O}_2, \ldots$ of X there are finite $\mathcal{F}_1 \subseteq \mathcal{O}_1, \mathcal{F}_2 \subseteq \mathcal{O}_2, \ldots$ such that $\mathcal{F}_1 \cup \mathcal{F}_2 \cup \ldots$ covers X



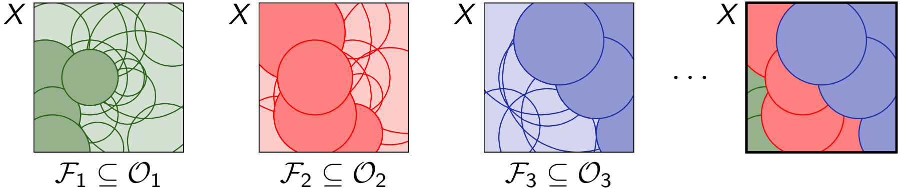
σ-compact $\Rightarrow$ Menger $\Rightarrow$ Lindelöf

Aurichi: Every Menger space is D

Chodunsky, Repovš, Zdomskyy: Mengers property characterizes filters whose Mathias forcing notion does not add dominating functions

The Menger property

Menger's property: for every sequence of open covers $\mathcal{O}_1, \mathcal{O}_2, \ldots$ of X there are finite $\mathcal{F}_1 \subseteq \mathcal{O}_1, \mathcal{F}_2 \subseteq \mathcal{O}_2, \ldots$ such that $\mathcal{F}_1 \cup \mathcal{F}_2 \cup \ldots$ covers X



σ-compact $\Rightarrow$ Menger $\Rightarrow$ Lindelöf

Aurichi: Every Menger space is D

Chodunsky, Repovš, Zdomskyy: Mengers property characterizes filters whose Mathias forcing notion does not add dominating functions

Tsaban: The most general class for which a general form of Hindmans Finite Sums Theorem holds

Menger meets combinatorics

$[\mathbb{N}]^\infty$: infinite subsets of $\mathbb{N}$

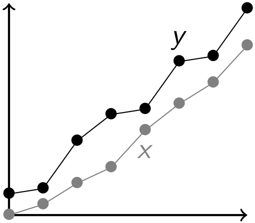
$[\mathbb{N}]^\infty \ni x = \{x(1), x(2), \ldots\}$: increasing enumeration, $[\mathbb{N}]^\infty \subseteq \mathbb{N}^\mathbb{N}$

Menger meets combinatorics

$[\mathbb{N}]^\infty$: infinite subsets of $\mathbb{N}$

$[\mathbb{N}]^\infty \ni x = \{x(1), x(2), \dots\}$: increasing enumeration, $[\mathbb{N}]^\infty \subseteq \mathbb{N}^\mathbb{N}$

- $x \leq y$ if $x(n) \leq y(n)$ for all n

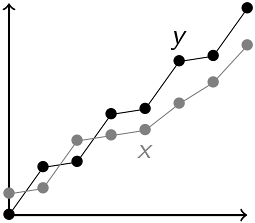


Menger meets combinatorics

$[\mathbb{N}]^\infty$: infinite subsets of $\mathbb{N}$

$[\mathbb{N}]^\infty \ni x = \{x(1), x(2), \ldots\}$: increasing enumeration, $[\mathbb{N}]^\infty \subseteq \mathbb{N}^\mathbb{N}$

- $x \leq y$ if $x(n) \leq y(n)$ for all n
- $x \leq^* d$ if $x(n) \leq y(n)$ for almost all n

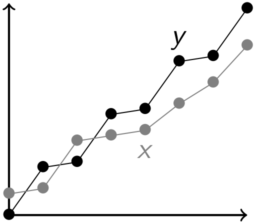


Menger meets combinatorics

$[\mathbb{N}]^\infty$: infinite subsets of $\mathbb{N}$

$[\mathbb{N}]^\infty \ni x = \{x(1), x(2), \ldots\}$: increasing enumeration, $[\mathbb{N}]^\infty \subseteq \mathbb{N}^\mathbb{N}$

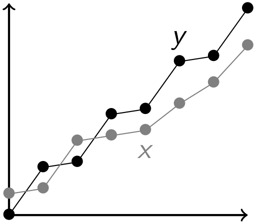
- $x \leq y$ if $x(n) \leq y(n)$ for all n
- $x \leq^* d$ if $x(n) \leq y(n)$ for almost all n
- Y is dominating if $\forall_{x \in [\mathbb{N}]^\infty} \ \exists_{y \in Y} \ x \leq^* y$



Menger meets combinatorics

$[\mathbb{N}]^\infty$: infinite subsets of $\mathbb{N}$

$[\mathbb{N}]^\infty \ni x = \{x(1), x(2), \ldots\}$: increasing enumeration, $[\mathbb{N}]^\infty \subseteq \mathbb{N}^\mathbb{N}$

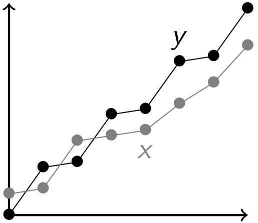


- $x \leq y$ if $x(n) \leq y(n)$ for all n
- $x \leq^* d$ if $x(n) \leq y(n)$ for almost all n
- Y is dominating if $\forall_{x \in [\mathbb{N}]^\infty} \exists_{y \in Y}\ x \leq^* y$
- $\mathfrak{d}$: minimal cardinality of a dominating set

Menger meets combinatorics

$[\mathbb{N}]^\infty$: infinite subsets of $\mathbb{N}$

$[\mathbb{N}]^\infty \ni x = \{x(1), x(2), \ldots\}$: increasing enumeration, $[\mathbb{N}]^\infty \subseteq \mathbb{N}^\mathbb{N}$



- $x \leq y$ if $x(n) \leq y(n)$ for all n
- $x \leq^* d$ if $x(n) \leq y(n)$ for almost all n
- Y is dominating if $\forall_{x \in [\mathbb{N}]^\infty} \exists_{y \in Y}\ x \leq^* y$
- $\mathfrak{d}$: minimal cardinality of a dominating set

Theorem (Hurewicz)

Assume that X is Lindelöf and zero-dimensional

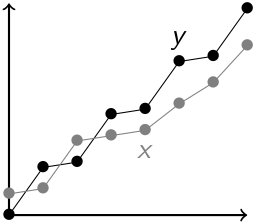
X is Menger $\Leftrightarrow$ continuous image of X into $[\mathbb{N}]^\infty$ is **nondominating**

Menger meets combinatorics

$[\mathbb{N}]^\infty$: infinite subsets of $\mathbb{N}$

$[\mathbb{N}]^\infty \ni x = \{x(1), x(2), \dots\}$: increasing enumeration, $[\mathbb{N}]^\infty \subseteq \mathbb{N}^\mathbb{N}$

- $x \leq y$ if $x(n) \leq y(n)$ for all n
- $x \leq^* d$ if $x(n) \leq y(n)$ for almost all n
- Y is dominating if $\forall_{x\in[\mathbb{N}]^\infty}\ \exists_{y\in Y}\ x \leq^* y$
- $\mathfrak{d}$: minimal cardinality of a dominating set



Theorem (Hurewicz)

Assume that X is Lindelöf and zero-dimensional

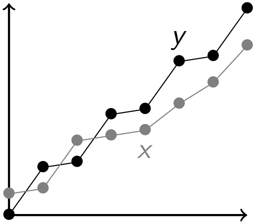
X is Menger $\Leftrightarrow$ continuous image of X into $[\mathbb{N}]^\infty$ is **nondominating**

- A Lindelöf X with $|X| < \mathfrak{d}$ is Menger

Menger meets combinatorics

$[\mathbb{N}]^\infty$: infinite subsets of $\mathbb{N}$

$[\mathbb{N}]^\infty \ni x = \{x(1), x(2), \dots\}$: increasing enumeration, $[\mathbb{N}]^\infty \subseteq \mathbb{N}^\mathbb{N}$



- $x \le y$ if $x(n) \le y(n)$ for all n
- $x \le^* d$ if $x(n) \le y(n)$ for almost all n
- Y is dominating if $\forall_{x \in [\mathbb{N}]^\infty}\ \exists_{y \in Y}\ x \le^* y$
- $\mathfrak{d}$: minimal cardinality of a dominating set

Theorem (Hurewicz)

Assume that X is Lindelöf and zero-dimensional
X is Menger $\Leftrightarrow$ continuous image of X into $[\mathbb{N}]^\infty$ is **nondominating**

- A Lindelöf X with $|X| < \mathfrak{d}$ is Menger
- A dominating $X \subseteq [\mathbb{N}]^\infty$ is not Menger

Products of Menger spaces

Products of Menger spaces

Todorčević (ZFC):

There is a Menger set whose square is not Menger

Products of Menger spaces

Todorčević (ZFC):

There is a Menger set whose square is not Menger

Sets of reals

Products of Menger spaces

Todorčević (ZFC):
There is a Menger set whose square is not Menger

Sets of reals

Just, Miller, Scheepers, Szeptycki (CH):
There is a Menger $M \subseteq \mathbb{R}$ whose square $M \times M$ is not Menger

Products of Menger spaces

Todorčević (ZFC):
There is a Menger set whose square is not Menger

Sets of reals

Just, Miller, Scheepers, Szeptycki (CH):
There is a Menger $M \subseteq \mathbb{R}$ whose square $M \times M$ is not Menger

Scheepers, Tsaban ($\mathrm{cov}(\mathcal{M}) = \mathrm{cof}(\mathcal{M})$):
There is a Menger $M \subseteq \mathbb{R}$ whose square $M \times M$ is not Menger

Products of Menger spaces

Todorčević (ZFC):

There is a Menger set whose square is not Menger

Sets of reals

Just, Miller, Scheepers, Szeptycki (CH):

There is a Menger $M \subseteq \mathbb{R}$ whose square $M \times M$ is not Menger

Scheepers, Tsaban ($\mathrm{cov}(\mathcal{M}) = \mathrm{cof}(\mathcal{M})$):

There is a Menger $M \subseteq \mathbb{R}$ whose square $M \times M$ is not Menger

Problem (sets of reals)

Find the minimal hypotheses that Menger's property is not productive

Products of Menger spaces

Todorčević (ZFC):
There is a Menger set whose square is not Menger

Sets of reals

Just, Miller, Scheepers, Szeptycki (CH):
There is a Menger $M \subseteq \mathbb{R}$ whose square $M \times M$ is not Menger

Scheepers, Tsaban ($\mathrm{cov}(\mathcal{M}) = \mathrm{cof}(\mathcal{M})$):
There is a Menger $M \subseteq \mathbb{R}$ whose square $M \times M$ is not Menger

Problem (sets of reals)

Find the minimal hypotheses that Menger's property is not productive

$\mathrm{P}(\mathbb{N}) \approx \{0,1\}^\omega$: the Cantor space

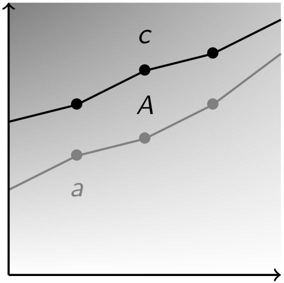
$\mathrm{P}(\mathbb{N}) = [\mathbb{N}]^\infty \cup \mathrm{Fin}$

$\mathfrak{d}$-unbounded sets

$A \subseteq [\mathbb{N}]^\infty$ is $\mathfrak{d}$-unbounded if $|A| \geq \mathfrak{d}$ and $\forall_{\mathbf{c} \in [\mathbb{N}]^\infty} |\{ a \in A : a \leq \mathbf{c} \}| < \mathfrak{d}$

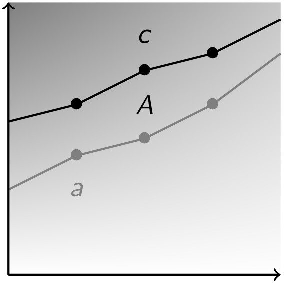
$\mathfrak{d}$-unbounded sets

$A \subseteq [\mathbb{N}]^\infty$ is $\mathfrak{d}$-unbounded if $|A| \geq \mathfrak{d}$ and $\forall_{\mathbf{c} \in [\mathbb{N}]^\infty} |\{ a \in A : a \leq \mathbf{c} \}| < \mathfrak{d}$



$\mathfrak{d}$-unbounded sets

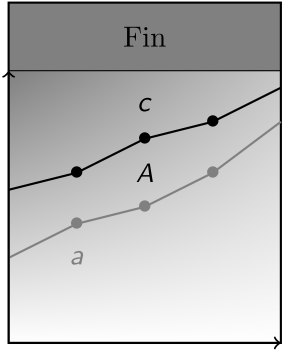
$A \subseteq [\mathbb{N}]^\infty$ is $\mathfrak{d}$-unbounded if $|A| \geq \mathfrak{d}$ and $\forall_{\mathbf{c} \in [\mathbb{N}]^\infty} |\{ a \in A : a \leq \mathbf{c} \}| < \mathfrak{d}$



$\mathfrak{d}$-unbounded sets

$A \subseteq [\mathbb{N}]^\infty$ is $\mathfrak{d}$-unbounded if $|A| \geq \mathfrak{d}$ and $\forall_{\mathbf{c} \in [\mathbb{N}]^\infty} |\{ a \in A : a \leq \mathbf{c} \}| < \mathfrak{d}$

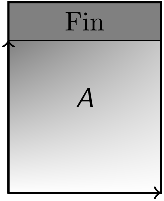
$$A \subseteq [\mathbb{N}]^\infty \text{ is } \mathfrak{d}\text{-unbounded} \Rightarrow A \cup \mathrm{Fin} \text{ is Menger}$$



$\mathfrak{d}$-unbounded sets

$A \subseteq [\mathbb{N}]^\infty$ is $\mathfrak{d}$-unbounded if $|A| \geq \mathfrak{d}$ and $\forall_{\mathbf{c} \in [\mathbb{N}]^\infty} |\{ a \in A : a \leq \mathbf{c} \}| < \mathfrak{d}$

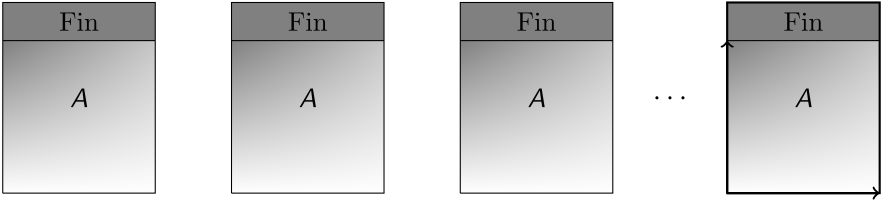
$A \subseteq [\mathbb{N}]^\infty$ is $\mathfrak{d}$-unbounded $\Rightarrow$ $A \cup \mathrm{Fin}$ is Menger



$\mathfrak{d}$-unbounded sets

$A \subseteq [\mathbb{N}]^\infty$ is $\mathfrak{d}$-unbounded if $|A| \geq \mathfrak{d}$ and $\forall_{\mathbf{c} \in [\mathbb{N}]^\infty} |\{ a \in A : a \leq \mathbf{c} \}| < \mathfrak{d}$

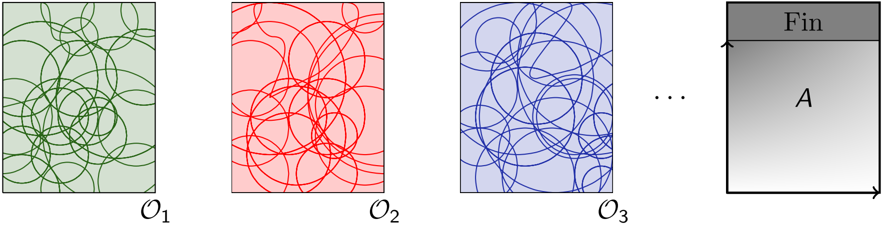
$A \subseteq [\mathbb{N}]^\infty$ is $\mathfrak{d}$-unbounded $\Rightarrow$ $A \cup \mathrm{Fin}$ is Menger



$\mathfrak{d}$-unbounded sets

$A \subseteq [\mathbb{N}]^\infty$ is $\mathfrak{d}$-unbounded if $|A| \geq \mathfrak{d}$ and $\forall_{\mathbf{c} \in [\mathbb{N}]^\infty} |\{\, a \in A : a \leq \mathbf{c} \,\}| < \mathfrak{d}$

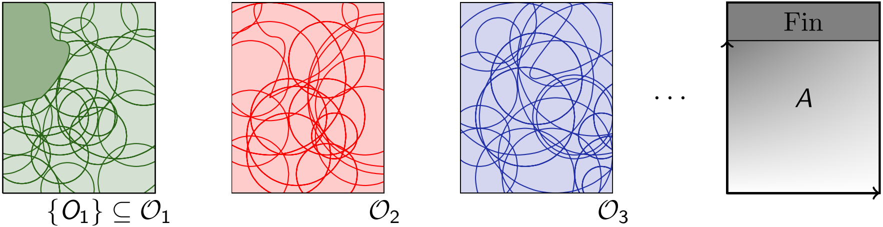
$A \subseteq [\mathbb{N}]^\infty$ is $\mathfrak{d}$-unbounded $\Rightarrow$ $A \cup \mathrm{Fin}$ is Menger



$\mathfrak{d}$-unbounded sets

$A \subseteq [\mathbb{N}]^\infty$ is $\mathfrak{d}$-unbounded if $|A| \geq \mathfrak{d}$ and $\forall_{\mathbf{c} \in [\mathbb{N}]^\infty} |\{ a \in A : a \leq \mathbf{c} \}| < \mathfrak{d}$

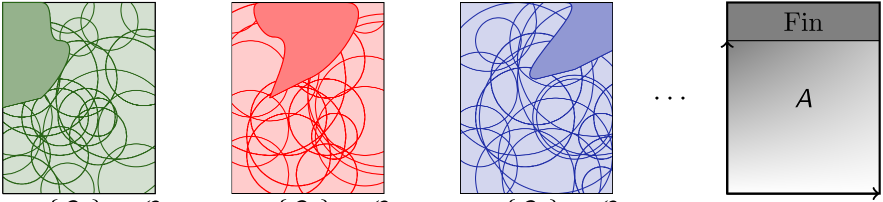
$A \subseteq [\mathbb{N}]^\infty$ is $\mathfrak{d}$-unbounded $\Rightarrow$ $A \cup \mathrm{Fin}$ is Menger



$\mathfrak{d}$-unbounded sets

$A \subseteq [\mathbb{N}]^\infty$ is $\mathfrak{d}$-unbounded if $|A| \geq \mathfrak{d}$ and $\forall_{\mathbf{c} \in [\mathbb{N}]^\infty} |\{ a \in A : a \leq \mathbf{c} \}| < \mathfrak{d}$

$$A \subseteq [\mathbb{N}]^\infty \text{ is } \mathfrak{d}\text{-unbounded} \Rightarrow A \cup \mathrm{Fin} \text{ is Menger}$$



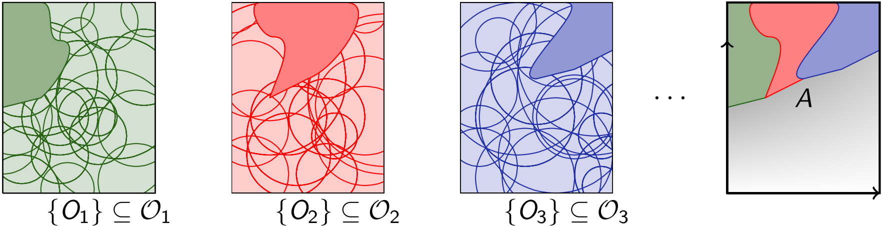
$\{O_1\} \subseteq \mathcal{O}_1$ $\{O_2\} \subseteq \mathcal{O}_2$ $\{O_3\} \subseteq \mathcal{O}_3$ $\cdots$

- $\mathrm{Fin} \subseteq \bigcup_n O_n$

$\mathfrak{d}$-unbounded sets

$A \subseteq [\mathbb{N}]^\infty$ is $\mathfrak{d}$-unbounded if $|A| \geq \mathfrak{d}$ and $\forall_{\mathbf{c} \in [\mathbb{N}]^\infty} |\{ a \in A : a \leq \mathbf{c} \}| < \mathfrak{d}$

$$A \subseteq [\mathbb{N}]^\infty \text{ is } \mathfrak{d}\text{-unbounded} \Rightarrow A \cup \mathrm{Fin} \text{ is Menger}$$

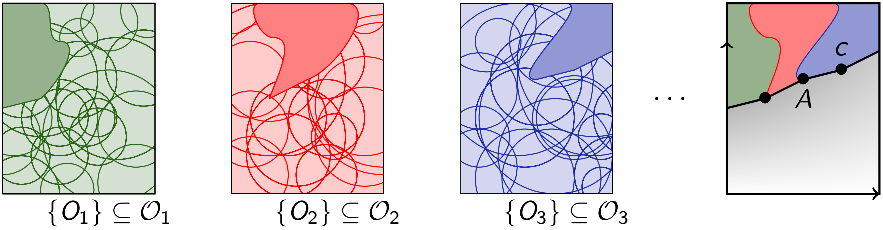


- $\mathrm{Fin} \subseteq \bigcup_n O_n$
- $\mathrm{P}(\mathbb{N}) \setminus \bigcup_n O_n \subseteq [\mathbb{N}]^\infty$ is compact

$\mathfrak{d}$-unbounded sets

$A \subseteq [\mathbb{N}]^\infty$ is $\mathfrak{d}$-unbounded if $|A| \geq \mathfrak{d}$ and $\forall_{\mathbf{c} \in [\mathbb{N}]^\infty} |\{ a \in A : a \leq \mathbf{c} \}| < \mathfrak{d}$

$$A \subseteq [\mathbb{N}]^\infty \text{ is } \mathfrak{d}\text{-unbounded} \Rightarrow A \cup \mathrm{Fin} \text{ is Menger}$$

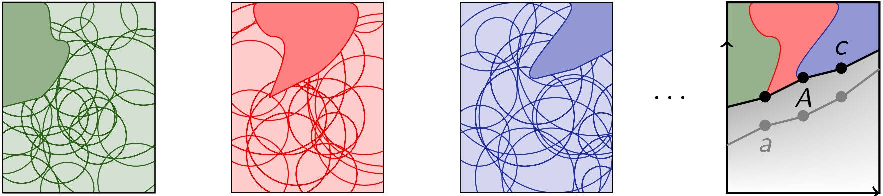


- $\mathrm{Fin} \subseteq \bigcup_n O_n$
- $\mathrm{P}(\mathbb{N}) \setminus \bigcup_n O_n \subseteq [\mathbb{N}]^\infty$ is compact, $\exists_{\mathbf{c} \in [\mathbb{N}]^\infty}\ \mathrm{P}(\mathbb{N}) \setminus \bigcup_n O_n \leq \mathbf{c}$

$\mathfrak{d}$-unbounded sets

$A \subseteq [\mathbb{N}]^\infty$ is $\mathfrak{d}$-unbounded if $|A| \geq \mathfrak{d}$ and $\forall_{\mathbf{c}\in[\mathbb{N}]^\infty} |\{\, a \in A : a \leq \mathbf{c} \,\}| < \mathfrak{d}$

$$A \subseteq [\mathbb{N}]^\infty \text{ is } \mathfrak{d}\text{-unbounded} \Rightarrow A \cup \mathrm{Fin} \text{ is Menger}$$



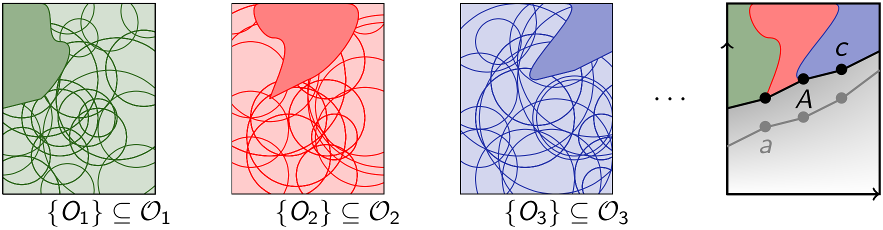
$\{O_1\} \subseteq \mathcal{O}_1$ $\{O_2\} \subseteq \mathcal{O}_2$ $\{O_3\} \subseteq \mathcal{O}_3$ $\dots$

- $\mathrm{Fin} \subseteq \bigcup_n O_n$
- $\mathrm{P}(\mathbb{N}) \setminus \bigcup_n O_n \subseteq [\mathbb{N}]^\infty$ is compact, $\exists_{\mathbf{c}\in[\mathbb{N}]^\infty}\ \mathrm{P}(\mathbb{N}) \setminus \bigcup_n O_n \leq \mathbf{c}$

$\mathfrak{d}$-unbounded sets

$A \subseteq [\mathbb{N}]^\infty$ is $\mathfrak{d}$-unbounded if $|A| \geq \mathfrak{d}$ and $\forall_{\mathbf{c} \in [\mathbb{N}]^\infty} |\{ a \in A : a \leq \mathbf{c} \}| < \mathfrak{d}$

$A \subseteq [\mathbb{N}]^\infty$ is $\mathfrak{d}$-unbounded $\Rightarrow$ $A \cup \mathrm{Fin}$ is Menger

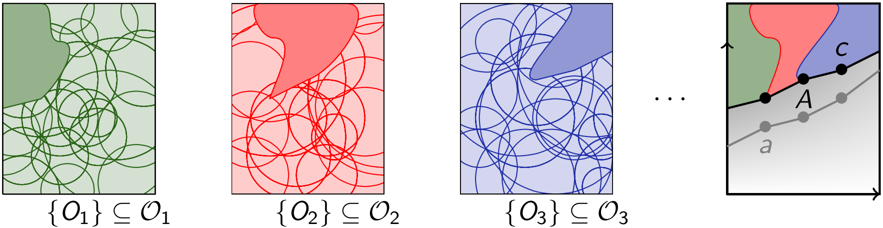


- $\mathrm{Fin} \subseteq \bigcup_n O_n$
- $\mathrm{P}(\mathbb{N}) \setminus \bigcup_n O_n \subseteq [\mathbb{N}]^\infty$ is compact, $\exists_{\mathbf{c} \in [\mathbb{N}]^\infty}$ $\mathrm{P}(\mathbb{N}) \setminus \bigcup_n O_n \leq \mathbf{c}$
- $|A \setminus \bigcup_n O_n| < \mathfrak{d}$

$\mathfrak{d}$-unbounded sets

$A \subseteq [\mathbb{N}]^\infty$ is $\mathfrak{d}$-unbounded if $|A| \geq \mathfrak{d}$ and $\forall_{\mathbf{c} \in [\mathbb{N}]^\infty} |\{\, a \in A : a \leq \mathbf{c} \,\}| < \mathfrak{d}$

$$A \subseteq [\mathbb{N}]^\infty \text{ is } \mathfrak{d}\text{-unbounded} \Rightarrow A \cup \mathrm{Fin} \text{ is Menger}$$

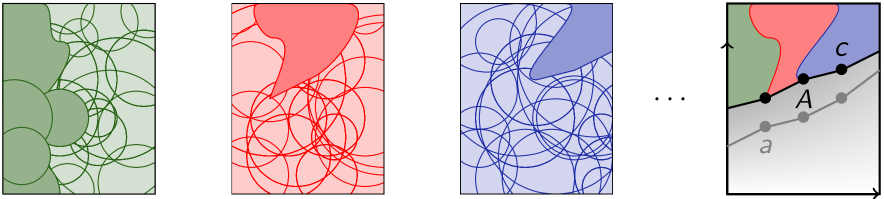


- $\mathrm{Fin} \subseteq \bigcup_n O_n$
- $\mathrm{P}(\mathbb{N}) \setminus \bigcup_n O_n \subseteq [\mathbb{N}]^\infty$ is compact, $\exists_{\mathbf{c} \in [\mathbb{N}]^\infty}\ \mathrm{P}(\mathbb{N}) \setminus \bigcup_n O_n \leq \mathbf{c}$
- $|A \setminus \bigcup_n O_n| < \mathfrak{d} \Rightarrow A \setminus \bigcup_n O_n$ is Menger

$\mathfrak{d}$-unbounded sets

$A \subseteq [\mathbb{N}]^\infty$ is $\mathfrak{d}$-unbounded if $|A| \geq \mathfrak{d}$ and $\forall_{\mathbf{c} \in [\mathbb{N}]^\infty} |\{\, a \in A : a \leq \mathbf{c} \,\}| < \mathfrak{d}$

$$A \subseteq [\mathbb{N}]^\infty \text{ is } \mathfrak{d}\text{-unbounded} \Rightarrow A \cup \mathrm{Fin} \text{ is Menger}$$



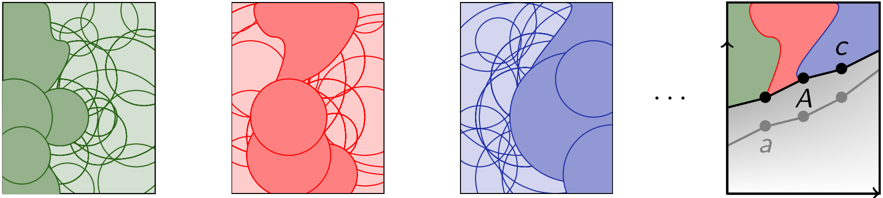
$\mathcal{F}_1 \cup \{O_1\} \subseteq \mathcal{O}_1$ $\quad$ $\{O_2\} \subseteq \mathcal{O}_2$ $\quad$ $\{O_3\} \subseteq \mathcal{O}_3$

- $\mathrm{Fin} \subseteq \bigcup_n O_n$
- $\mathrm{P}(\mathbb{N}) \setminus \bigcup_n O_n \subseteq [\mathbb{N}]^\infty$ is compact, $\exists_{\mathbf{c} \in [\mathbb{N}]^\infty}\ \mathrm{P}(\mathbb{N}) \setminus \bigcup_n O_n \leq \mathbf{c}$
- $|A \setminus \bigcup_n O_n| < \mathfrak{d} \Rightarrow A \setminus \bigcup_n O_n$ is Menger

$\mathfrak{d}$-unbounded sets

$A \subseteq [\mathbb{N}]^\infty$ is $\mathfrak{d}$-unbounded if $|A| \geq \mathfrak{d}$ and $\forall_{\mathbf{c} \in [\mathbb{N}]^\infty} |\{ a \in A : a \leq \mathbf{c} \}| < \mathfrak{d}$

$$A \subseteq [\mathbb{N}]^\infty \text{ is } \mathfrak{d}\text{-unbounded} \Rightarrow A \cup \mathrm{Fin} \text{ is Menger}$$



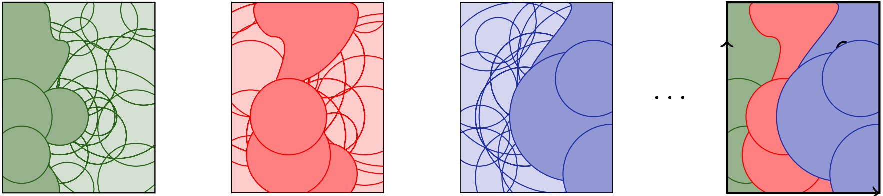
$\mathcal{F}_1 \cup \{O_1\} \subseteq \mathcal{O}_1$ $\quad$ $\mathcal{F}_2 \cup \{O_2\} \subseteq \mathcal{O}_2$ $\quad$ $\mathcal{F}_3 \cup \{O_3\} \subseteq \mathcal{O}_3$

- $\mathrm{Fin} \subseteq \bigcup_n O_n$
- $\mathrm{P}(\mathbb{N}) \setminus \bigcup_n O_n \subseteq [\mathbb{N}]^\infty$ is compact, $\exists_{\mathbf{c} \in [\mathbb{N}]^\infty}\ \mathrm{P}(\mathbb{N}) \setminus \bigcup_n O_n \leq \mathbf{c}$
- $|A \setminus \bigcup_n O_n| < \mathfrak{d} \Rightarrow A \setminus \bigcup_n O_n$ is Menger

$\mathfrak{d}$-unbounded sets

$A \subseteq [\mathbb{N}]^\infty$ is $\mathfrak{d}$-unbounded if $|A| \geq \mathfrak{d}$ and $\forall_{\mathbf{c}\in[\mathbb{N}]^\infty} |\{\, a \in A : a \leq \mathbf{c} \,\}| < \mathfrak{d}$

$A \subseteq [\mathbb{N}]^\infty$ is $\mathfrak{d}$-unbounded $\Rightarrow$ $A \cup \mathrm{Fin}$ is Menger



$\mathcal{F}_1 \cup \{O_1\} \subseteq \mathcal{O}_1$ $\mathcal{F}_2 \cup \{O_2\} \subseteq \mathcal{O}_2$ $\mathcal{F}_3 \cup \{O_3\} \subseteq \mathcal{O}_3$ $\dots$

- $\mathrm{Fin} \subseteq \bigcup_n O_n$
- $\mathrm{P}(\mathbb{N}) \setminus \bigcup_n O_n \subseteq [\mathbb{N}]^\infty$ is compact, $\exists_{\mathbf{c}\in[\mathbb{N}]^\infty}$ $\mathrm{P}(\mathbb{N}) \setminus \bigcup_n O_n \leq \mathbf{c}$
- $|A \setminus \bigcup_n O_n| < \mathfrak{d} \Rightarrow A \setminus \bigcup_n O_n$ is Menger
- $A \cup \mathrm{Fin}$ is Menger

Main results

$A \subseteq [\mathbb{N}]^\infty$ is $\mathfrak{d}$-unbounded if $|A| \geq \mathfrak{d}$ and $\forall_{\mathbf{c} \in [\mathbb{N}]^\infty} |\{ a \in A : a \leq \mathbf{c} \}| < \mathfrak{d}$

Theorem (Sz, Tsaban)

If $X \subseteq [\mathbb{N}]^\infty$ contains a $\mathfrak{d}$-unbounded set or a $\mathrm{cf}(\mathfrak{d})$-unbounded set, then there is a Menger $Y \subseteq \mathrm{P}(\mathbb{N})$, $X \times Y$ is not Menger

Main results

$A \subseteq [\mathbb{N}]^\infty$ is $\mathfrak{d}$-unbounded if $|A| \geq \mathfrak{d}$ and $\forall_{\mathbf{c} \in [\mathbb{N}]^\infty} |\{ a \in A : a \leq \mathbf{c} \}| < \mathfrak{d}$

Theorem (Sz, Tsaban)

If $X \subseteq [\mathbb{N}]^\infty$ contains a $\mathfrak{d}$-unbounded set or a $\mathrm{cf}(\mathfrak{d})$-unbounded set, then there is a Menger $Y \subseteq \mathrm{P}(\mathbb{N})$, $X \times Y$ is not Menger

Corollary

$\mathrm{cf}(\mathfrak{d}) < \mathfrak{d} \Rightarrow \exists$ *Menger* $X, Y \subseteq \mathrm{P}(\mathbb{N})$, $X \times Y$ *is not Menger*

Main results

$A \subseteq [\mathbb{N}]^\infty$ is $\mathfrak{d}$-unbounded if $|A| \geq \mathfrak{d}$ and $\forall_{\mathbf{c} \in [\mathbb{N}]^\infty} |\{ a \in A : a \leq \mathbf{c} \}| < \mathfrak{d}$

Theorem (Sz, Tsaban)

If $X \subseteq [\mathbb{N}]^\infty$ contains a $\mathfrak{d}$-unbounded set or a $\mathrm{cf}(\mathfrak{d})$-unbounded set, then there is a Menger $Y \subseteq \mathrm{P}(\mathbb{N})$, $X \times Y$ is not Menger

Corollary

$\mathrm{cf}(\mathfrak{d}) < \mathfrak{d} \Rightarrow \exists$ *Menger* $X, Y \subseteq \mathrm{P}(\mathbb{N})$, $X \times Y$ *is not Menger*

- $\exists$ $\mathrm{cf}(\mathfrak{d})$-unbounded $X \subseteq [\mathbb{N}]^\infty$, $|X| = \mathrm{cf}(\mathfrak{d})$

Main results

$A \subseteq [\mathbb{N}]^\infty$ is $\mathfrak{d}$-unbounded if $|A| \geq \mathfrak{d}$ and $\forall_{\mathbf{c} \in [\mathbb{N}]^\infty} |\{ a \in A : a \leq \mathbf{c} \}| < \mathfrak{d}$

Theorem (Sz, Tsaban)

If $X \subseteq [\mathbb{N}]^\infty$ contains a $\mathfrak{d}$-unbounded set or a $\mathrm{cf}(\mathfrak{d})$-unbounded set, then there is a Menger $Y \subseteq \mathrm{P}(\mathbb{N})$, $X \times Y$ is not Menger

Corollary

$\mathrm{cf}(\mathfrak{d}) < \mathfrak{d} \Rightarrow \exists$ *Menger* $X, Y \subseteq \mathrm{P}(\mathbb{N})$, $X \times Y$ *is not Menger*

- $\exists$ $\mathrm{cf}(\mathfrak{d})$-unbounded $X \subseteq [\mathbb{N}]^\infty$, $|X| = \mathrm{cf}(\mathfrak{d})$
- $|X| = \mathrm{cf}(\mathfrak{d}) < \mathfrak{d} \Rightarrow X$ is Menger

Main results

$A \subseteq [\mathbb{N}]^\infty$ is $\mathfrak{d}$-unbounded if $|A| \geq \mathfrak{d}$ and $\forall_{\mathbf{c} \in [\mathbb{N}]^\infty} |\{ a \in A : a \leq \mathbf{c} \}| < \mathfrak{d}$

Theorem (Sz, Tsaban)

If $X \subseteq [\mathbb{N}]^\infty$ contains a $\mathfrak{d}$-unbounded set or a $\mathrm{cf}(\mathfrak{d})$-unbounded set, then there is a Menger $Y \subseteq \mathrm{P}(\mathbb{N})$, $X \times Y$ is not Menger

Corollary

$$\mathrm{cf}(\mathfrak{d}) < \mathfrak{d} \Rightarrow \exists \textit{ Menger } X, Y \subseteq \mathrm{P}(\mathbb{N}),\ X \times Y \textit{ is not Menger}$$

- $\exists$ $\mathrm{cf}(\mathfrak{d})$-unbounded $X \subseteq [\mathbb{N}]^\infty$, $|X| = \mathrm{cf}(\mathfrak{d})$
- $|X| = \mathrm{cf}(\mathfrak{d}) < \mathfrak{d} \Rightarrow X$ is Menger
- $\exists$ Menger $Y \subseteq [\mathbb{N}]^\infty$, $X \times Y$ is not Menger

Main results

$A \subseteq [\mathbb{N}]^\infty$ is $\mathfrak{d}$-unbounded if $|A| \geq \mathfrak{d}$ and $\forall_{\mathbf{c} \in [\mathbb{N}]^\infty} |\{\, a \in A : a \leq \mathbf{c} \,\}| < \mathfrak{d}$

Main results

$A \subseteq [\mathbb{N}]^\infty$ is $\mathfrak{d}$-unbounded if $|A| \geq \mathfrak{d}$ and $\forall_{\mathbf{c} \in [\mathbb{N}]^\infty} |\{ a \in A : a \leq \mathbf{c} \}| < \mathfrak{d}$

$A \subseteq [\mathbb{N}]^{\infty,\infty}$ is bi-$\mathfrak{d}$-unbounded if A and $\{ a^c : a \in A \}$ are $\mathfrak{d}$-unbounded

Main results

$A \subseteq [\mathbb{N}]^\infty$ is $\mathfrak{d}$-unbounded if $|A| \geq \mathfrak{d}$ and $\forall_{\mathbf{c} \in [\mathbb{N}]^\infty} |\{ a \in A : a \leq \mathbf{c} \}| < \mathfrak{d}$

$A \subseteq [\mathbb{N}]^{\infty,\infty}$ is bi-$\mathfrak{d}$-unbounded if A and $\{ a^c : a \in A \}$ are $\mathfrak{d}$-unbounded

$\mathfrak{r}$: min card of $A \subseteq [\mathbb{N}]^\infty$, there is no $r \in [\mathbb{N}]^\infty$ s.t. for all $a \in A$
$r \cap a$ and $r \setminus a$ are infinite

Main results

$A \subseteq [\mathbb{N}]^\infty$ is $\mathfrak{d}$-unbounded if $|A| \geq \mathfrak{d}$ and $\forall_{\mathbf{c} \in [\mathbb{N}]^\infty} |\{ a \in A : a \leq \mathbf{c} \}| < \mathfrak{d}$

$A \subseteq [\mathbb{N}]^{\infty,\infty}$ is bi-$\mathfrak{d}$-unbounded if A and $\{ a^c : a \in A \}$ are $\mathfrak{d}$-unbounded

$\mathfrak{r}$: min card of $A \subseteq [\mathbb{N}]^\infty$, there is no $r \in [\mathbb{N}]^\infty$ s.t. for all $a \in A$
$r \cap a$ and $r \setminus a$ are infinite

Corollary

$\mathfrak{d} \leq \mathfrak{r} \Rightarrow \exists$ *Menger* $X, Y \subseteq \mathrm{P}(\mathbb{N})$, $X \times Y$ *is not Menger*

Main results

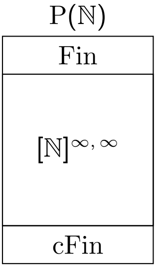
$A \subseteq [\mathbb{N}]^\infty$ is $\mathfrak{d}$-unbounded if $|A| \geq \mathfrak{d}$ and $\forall_{\mathbf{c} \in [\mathbb{N}]^\infty} |\{ a \in A : a \leq \mathbf{c} \}| < \mathfrak{d}$

$A \subseteq [\mathbb{N}]^{\infty,\infty}$ is bi-$\mathfrak{d}$-unbounded if A and $\{ a^c : a \in A \}$ are $\mathfrak{d}$-unbounded

$\mathfrak{r}$: min card of $A \subseteq [\mathbb{N}]^\infty$, there is no $r \in [\mathbb{N}]^\infty$ s.t. for all $a \in A$ $r \cap a$ and $r \setminus a$ are infinite

Corollary

$\mathfrak{d} \leq \mathfrak{r} \Rightarrow \exists$ *Menger* $X, Y \subseteq \mathrm{P}(\mathbb{N})$, $X \times Y$ *is not Menger*



Main results

$A \subseteq [\mathbb{N}]^\infty$ is $\mathfrak{d}$-unbounded if $|A| \geq \mathfrak{d}$ and $\forall_{\mathbf{c} \in [\mathbb{N}]^\infty} |\{ a \in A : a \leq \mathbf{c} \}| < \mathfrak{d}$

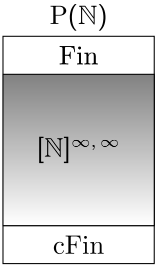
$A \subseteq [\mathbb{N}]^{\infty,\infty}$ is bi-$\mathfrak{d}$-unbounded if A and $\{ a^c : a \in A \}$ are $\mathfrak{d}$-unbounded

$\mathfrak{r}$: min card of $A \subseteq [\mathbb{N}]^\infty$, there is no $r \in [\mathbb{N}]^\infty$ s.t. for all $a \in A$
$r \cap a$ and $r \setminus a$ are infinite

Corollary

$\mathfrak{d} \leq \mathfrak{r} \Rightarrow \exists$ *Menger* $X, Y \subseteq \mathrm{P}(\mathbb{N})$, $X \times Y$ *is not Menger*

- $\mathfrak{d} \leq \mathfrak{r} \Leftrightarrow \exists$ bi-$\mathfrak{d}$-unbounded $A \subseteq [\mathbb{N}]^{\infty,\infty}$



Main results

$A \subseteq [\mathbb{N}]^\infty$ is $\mathfrak{d}$-unbounded if $|A| \geq \mathfrak{d}$ and $\forall_{\mathbf{c} \in [\mathbb{N}]^\infty} |\{ a \in A : a \leq \mathbf{c} \}| < \mathfrak{d}$

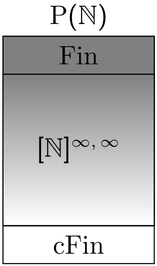
$A \subseteq [\mathbb{N}]^{\infty,\infty}$ is bi-$\mathfrak{d}$-unbounded if A and $\{ a^c : a \in A \}$ are $\mathfrak{d}$-unbounded

$\mathfrak{r}$: min card of $A \subseteq [\mathbb{N}]^\infty$, there is no $r \in [\mathbb{N}]^\infty$ s.t. for all $a \in A$ $r \cap a$ and $r \setminus a$ are infinite

Corollary

$\mathfrak{d} \leq \mathfrak{r} \Rightarrow \exists$ *Menger* $X, Y \subseteq \mathrm{P}(\mathbb{N})$, $X \times Y$ *is not Menger*

- $\mathfrak{d} \leq \mathfrak{r} \Leftrightarrow \exists$ bi-$\mathfrak{d}$-unbounded $A \subseteq [\mathbb{N}]^{\infty,\infty}$
- $A \cup \mathrm{Fin}$ is Menger



Main results

$A \subseteq [\mathbb{N}]^\infty$ is $\mathfrak{d}$-unbounded if $|A| \geq \mathfrak{d}$ and $\forall_{\mathbf{c} \in [\mathbb{N}]^\infty} |\{ a \in A : a \leq \mathbf{c} \}| < \mathfrak{d}$

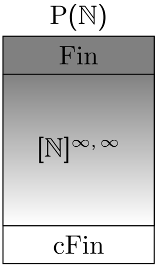
$A \subseteq [\mathbb{N}]^{\infty,\infty}$ is bi-$\mathfrak{d}$-unbounded if A and $\{ a^c : a \in A \}$ are $\mathfrak{d}$-unbounded

$\mathfrak{r}$: min card of $A \subseteq [\mathbb{N}]^\infty$, there is no $r \in [\mathbb{N}]^\infty$ s.t. for all $a \in A$
$r \cap a$ and $r \setminus a$ are infinite

Corollary

$\mathfrak{d} \leq \mathfrak{r} \Rightarrow \exists$ *Menger* $X, Y \subseteq \mathrm{P}(\mathbb{N})$, $X \times Y$ *is not Menger*

- $\mathfrak{d} \leq \mathfrak{r} \Leftrightarrow \exists$ bi-$\mathfrak{d}$-unbounded $A \subseteq [\mathbb{N}]^{\infty,\infty}$
- $A \cup \mathrm{Fin}$ is Menger
- $\tau : \mathrm{P}(\mathbb{N}) \to \mathrm{P}(\mathbb{N})$, $\tau(a) = a^c = a \oplus \mathbb{N}$



Main results

$A \subseteq [\mathbb{N}]^\infty$ is $\mathfrak{d}$-unbounded if $|A| \geq \mathfrak{d}$ and $\forall_{\mathbf{c}\in[\mathbb{N}]^\infty} |\{ a \in A : a \leq \mathbf{c} \}| < \mathfrak{d}$

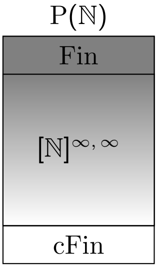
$A \subseteq [\mathbb{N}]^{\infty,\infty}$ is bi-$\mathfrak{d}$-unbounded if A and $\{ a^c : a \in A \}$ are $\mathfrak{d}$-unbounded

$\mathfrak{r}$: min card of $A \subseteq [\mathbb{N}]^\infty$, there is no $r \in [\mathbb{N}]^\infty$ s.t. for all $a \in A$ $r \cap a$ and $r \setminus a$ are infinite

Corollary

$\mathfrak{d} \leq \mathfrak{r} \Rightarrow \exists$ *Menger* $X, Y \subseteq \mathrm{P}(\mathbb{N})$, $X \times Y$ *is not Menger*

- $\mathfrak{d} \leq \mathfrak{r} \Leftrightarrow \exists$ bi-$\mathfrak{d}$-unbounded $A \subseteq [\mathbb{N}]^{\infty,\infty}$
- $A \cup \mathrm{Fin}$ is Menger
- $\tau : \mathrm{P}(\mathbb{N}) \to \mathrm{P}(\mathbb{N})$, $\tau(a) = a^c = a \oplus \mathbb{N}$
- $X = \tau[A \cup \mathrm{Fin}] = \{ a^c : a \in A \} \cup \mathrm{cFin} \subseteq [\mathbb{N}]^\infty$



Main results

$A \subseteq [\mathbb{N}]^\infty$ is $\mathfrak{d}$-unbounded if $|A| \geq \mathfrak{d}$ and $\forall_{\mathbf{c}\in[\mathbb{N}]^\infty} |\{\, a \in A : a \leq \mathbf{c}\,\}| < \mathfrak{d}$

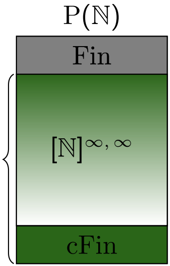
$A \subseteq [\mathbb{N}]^{\infty,\infty}$ is bi-$\mathfrak{d}$-unbounded if A and $\{\, a^c : a \in A\,\}$ are $\mathfrak{d}$-unbounded

$\mathfrak{r}$: min card of $A \subseteq [\mathbb{N}]^\infty$, there is no $r \in [\mathbb{N}]^\infty$ s.t. for all $a \in A$ $r \cap a$ and $r \setminus a$ are infinite

Corollary

$\mathfrak{d} \leq \mathfrak{r} \Rightarrow \exists$ *Menger* $X, Y \subseteq \mathrm{P}(\mathbb{N})$, $X \times Y$ *is not Menger*

- $\mathfrak{d} \leq \mathfrak{r} \Leftrightarrow \exists$ bi-$\mathfrak{d}$-unbounded $A \subseteq [\mathbb{N}]^{\infty,\infty}$
- $A \cup \mathrm{Fin}$ is Menger
- $\tau\colon \mathrm{P}(\mathbb{N}) \to \mathrm{P}(\mathbb{N})$, $\tau(a) = a^c = a \oplus \mathbb{N}$
- $X = \tau[A \cup \mathrm{Fin}] = \{\, a^c : a \in A\,\} \cup \mathrm{cFin} \subseteq [\mathbb{N}]^\infty$



Main results

$A \subseteq [\mathbb{N}]^\infty$ is $\mathfrak{d}$-unbounded if $|A| \geq \mathfrak{d}$ and $\forall_{\mathbf{c} \in [\mathbb{N}]^\infty} |\{ a \in A : a \leq \mathbf{c} \}| < \mathfrak{d}$

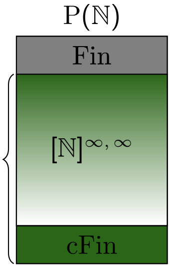
$A \subseteq [\mathbb{N}]^{\infty,\infty}$ is bi-$\mathfrak{d}$-unbounded if A and $\{ a^c : a \in A \}$ are $\mathfrak{d}$-unbounded

$\mathfrak{r}$: min card of $A \subseteq [\mathbb{N}]^\infty$, there is no $r \in [\mathbb{N}]^\infty$ s.t. for all $a \in A$
$r \cap a$ and $r \setminus a$ are infinite

Corollary

$\mathfrak{d} \leq \mathfrak{r} \Rightarrow \exists$ Menger $X, Y \subseteq \mathrm{P}(\mathbb{N})$, $X \times Y$ is not Menger

- $\mathfrak{d} \leq \mathfrak{r} \Leftrightarrow \exists$ bi-$\mathfrak{d}$-unbounded $A \subseteq [\mathbb{N}]^{\infty,\infty}$
- $A \cup \mathrm{Fin}$ is Menger
- $\tau : \mathrm{P}(\mathbb{N}) \to \mathrm{P}(\mathbb{N})$, $\tau(a) = a^c = a \oplus \mathbb{N}$
- $X = \tau[A \cup \mathrm{Fin}] = \{ a^c : a \in A \} \cup \mathrm{cFin} \subseteq [\mathbb{N}]^\infty$
- $\mathfrak{d}$-unbounded $\{ a^c : a \in A \} \subseteq X$



Main results

$A \subseteq [\mathbb{N}]^\infty$ is $\mathfrak{d}$-unbounded if $|A| \geq \mathfrak{d}$ and $\forall_{\mathbf{c} \in [\mathbb{N}]^\infty} |\{ a \in A : a \leq \mathbf{c} \}| < \mathfrak{d}$

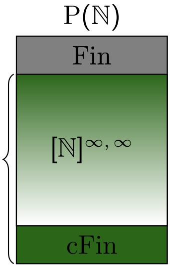
$A \subseteq [\mathbb{N}]^{\infty,\infty}$ is bi-$\mathfrak{d}$-unbounded if A and $\{ a^c : a \in A \}$ are $\mathfrak{d}$-unbounded

$\mathfrak{r}$: min card of $A \subseteq [\mathbb{N}]^\infty$, there is no $r \in [\mathbb{N}]^\infty$ s.t. for all $a \in A$
$r \cap a$ and $r \setminus a$ are infinite

Corollary

$\mathfrak{d} \leq \mathfrak{r} \Rightarrow \exists$ *Menger* $X, Y \subseteq \mathrm{P}(\mathbb{N})$, $X \times Y$ *is not Menger*

- $\mathfrak{d} \leq \mathfrak{r} \Leftrightarrow \exists$ bi-$\mathfrak{d}$-unbounded $A \subseteq [\mathbb{N}]^{\infty,\infty}$
- $A \cup \mathrm{Fin}$ is Menger
- $\tau : \mathrm{P}(\mathbb{N}) \to \mathrm{P}(\mathbb{N})$, $\tau(a) = a^c = a \oplus \mathbb{N}$
- $X = \tau[A \cup \mathrm{Fin}] = \{ a^c : a \in A \} \cup \mathrm{cFin} \subseteq [\mathbb{N}]^\infty$
- $\mathfrak{d}$-unbounded $\{ a^c : a \in A \} \subseteq X$
- $\exists$ Menger $Y \subseteq \mathrm{P}(\mathbb{N})$, $X \times Y$ is not Menger



Main results

$A \subseteq [\mathbb{N}]^\infty$ is $\mathfrak{d}$-unbounded if $|A| \geq \mathfrak{d}$ and $\forall_{\mathbf{c} \in [\mathbb{N}]^\infty} |\{ a \in A : a \leq \mathbf{c} \}| < \mathfrak{d}$

$A \subseteq [\mathbb{N}]^{\infty,\infty}$ is bi-$\mathfrak{d}$-unbounded if A and $\{ a^c : a \in A \}$ are $\mathfrak{d}$-unbounded

$\mathfrak{r}$: min card of $A \subseteq [\mathbb{N}]^\infty$, there is no $r \in [\mathbb{N}]^\infty$ s.t. for all $a \in A$
$r \cap a$ and $r \setminus a$ are infinite

Corollary

$\mathfrak{d} \leq \mathfrak{r} \Rightarrow \exists$ *Menger* $X, Y \subseteq \mathrm{P}(\mathbb{N})$, $X \times Y$ *is not Menger*

Productivity of Menger

MA	Cohen	Random	Sacks	Hechler	Laver	Mathias	Miller

Main results

$A \subseteq [\mathbb{N}]^\infty$ is $\mathfrak{d}$-unbounded if $|A| \geq \mathfrak{d}$ and $\forall_{\mathbf{c} \in [\mathbb{N}]^\infty} |\{ a \in A : a \leq \mathbf{c} \}| < \mathfrak{d}$

$A \subseteq [\mathbb{N}]^{\infty,\infty}$ is bi-$\mathfrak{d}$-unbounded if A and $\{ a^c : a \in A \}$ are $\mathfrak{d}$-unbounded

$\mathfrak{r}$: min card of $A \subseteq [\mathbb{N}]^\infty$, there is no $r \in [\mathbb{N}]^\infty$ s.t. for all $a \in A$ $r \cap a$ and $r \setminus a$ are infinite

Corollary

$\mathfrak{d} \leq \mathfrak{r} \Rightarrow \exists$ *Menger* $X, Y \subseteq \mathrm{P}(\mathbb{N})$, $X \times Y$ *is not Menger*

Productivity of Menger

MA	Cohen	Random	Sacks	Hechler	Laver	Mathias	Miller
✗	✗	✗	✗	✗	✗	✗	

Main results

$A \subseteq [\mathbb{N}]^\infty$ is $\mathfrak{d}$-unbounded if $|A| \geq \mathfrak{d}$ and $\forall_{\mathbf{c} \in [\mathbb{N}]^\infty} |\{ a \in A : a \leq \mathbf{c} \}| < \mathfrak{d}$

$A \subseteq [\mathbb{N}]^{\infty,\infty}$ is bi-$\mathfrak{d}$-unbounded if A and $\{ a^c : a \in A \}$ are $\mathfrak{d}$-unbounded

$\mathfrak{r}$: min card of $A \subseteq [\mathbb{N}]^\infty$, there is no $r \in [\mathbb{N}]^\infty$ s.t. for all $a \in A$
$r \cap a$ and $r \setminus a$ are infinite

Corollary

$\mathfrak{d} \leq \mathfrak{r} \Rightarrow \exists$ *Menger* $X, Y \subseteq \mathrm{P}(\mathbb{N})$, $X \times Y$ *is not Menger*

Productivity of Menger

MA	Cohen	Random	Sacks	Hechler	Laver	Mathias	Miller
✗	✗	✗	✗	✗	✗	✗	?

Main results

$A \subseteq [\mathbb{N}]^\infty$ is $\mathfrak{d}$-unbounded if $|A| \geq \mathfrak{d}$ and $\forall_{\mathbf{c} \in [\mathbb{N}]^\infty} |\{ a \in A : a \leq \mathbf{c} \}| < \mathfrak{d}$

$A \subseteq [\mathbb{N}]^{\infty,\infty}$ is bi-$\mathfrak{d}$-unbounded if A and $\{ a^c : a \in A \}$ are $\mathfrak{d}$-unbounded

$\mathfrak{r}$: min card of $A \subseteq [\mathbb{N}]^\infty$, there is no $r \in [\mathbb{N}]^\infty$ s.t. for all $a \in A$ $r \cap a$ and $r \setminus a$ are infinite

Corollary

$\mathfrak{d} \leq \mathfrak{r} \Rightarrow \exists$ *Menger* $X, Y \subseteq \mathrm{P}(\mathbb{N})$, $X \times Y$ *is not Menger*

Productivity of Menger

MA	Cohen	Random	Sacks	Hechler	Laver	Mathias	Miller
✗	✗	✗	✗	✗	✗	✗	✓?

Theorem? (Zdomskyy)

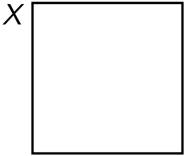
In the Miller model Menger is productive

The Hurewicz property

Hurewicz's property: for every sequence of open covers $\mathcal{O}_1, \mathcal{O}_2, \ldots$ of X there are finite $\mathcal{F}_1 \subseteq \mathcal{O}_1, \mathcal{F}_2 \subseteq \mathcal{O}_2, \ldots$ such that for each $x \in X$, the set $\{\, n \in \mathbb{N} : x \notin \bigcup \mathcal{F}_n \,\}$ is finite

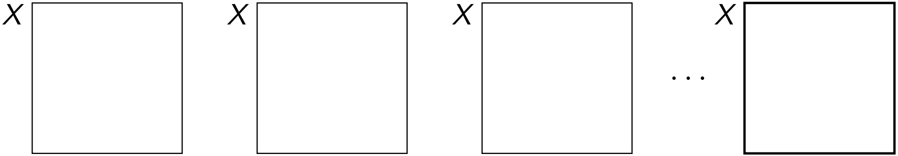
The Hurewicz property

Hurewicz's property: for every sequence of open covers $\mathcal{O}_1, \mathcal{O}_2, \ldots$ of X there are finite $\mathcal{F}_1 \subseteq \mathcal{O}_1, \mathcal{F}_2 \subseteq \mathcal{O}_2, \ldots$ such that for each $x \in X$, the set $\{ n \in \mathbb{N} : x \notin \bigcup \mathcal{F}_n \}$ is finite



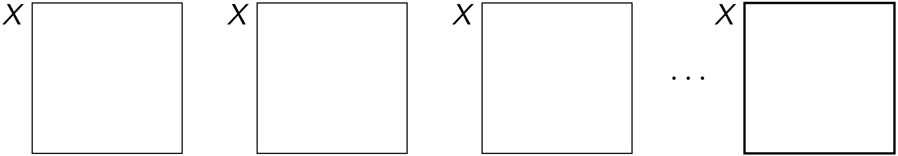
The Hurewicz property

Hurewicz's property: for every sequence of open covers $\mathcal{O}_1, \mathcal{O}_2, \ldots$ of X there are finite $\mathcal{F}_1 \subseteq \mathcal{O}_1, \mathcal{F}_2 \subseteq \mathcal{O}_2, \ldots$ such that for each $x \in X$, the set $\{\, n \in \mathbb{N} : x \notin \bigcup \mathcal{F}_n \,\}$ is finite



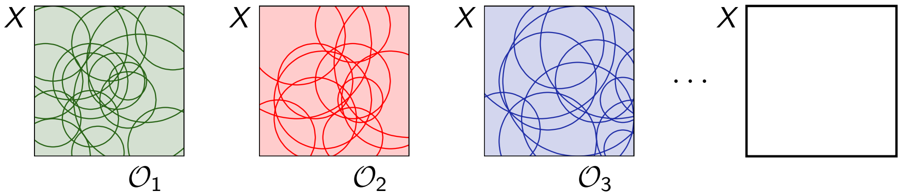
The Hurewicz property

Hurewicz's property: for every sequence of open covers $\mathcal{O}_1, \mathcal{O}_2, \ldots$ of X there are finite $\mathcal{F}_1 \subseteq \mathcal{O}_1, \mathcal{F}_2 \subseteq \mathcal{O}_2, \ldots$ such that for each $x \in X$, the set $\{\, n \in \mathbb{N} : x \notin \bigcup \mathcal{F}_n \,\}$ is finite



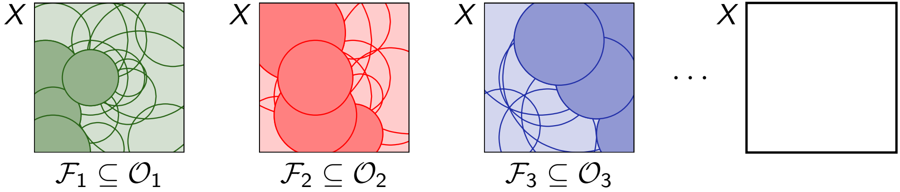
The Hurewicz property

Hurewicz's property: for every sequence of open covers $\mathcal{O}_1, \mathcal{O}_2, \ldots$ of X there are finite $\mathcal{F}_1 \subseteq \mathcal{O}_1, \mathcal{F}_2 \subseteq \mathcal{O}_2, \ldots$ such that for each $x \in X$, the set $\{ n \in \mathbb{N} : x \notin \bigcup \mathcal{F}_n \}$ is finite



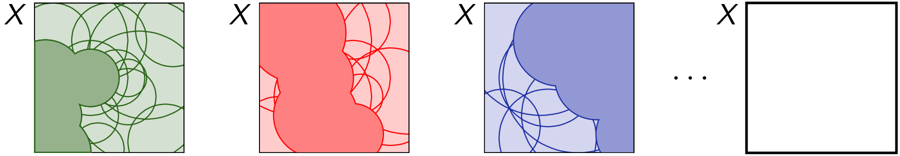
The Hurewicz property

Hurewicz's property: for every sequence of open covers $\mathcal{O}_1, \mathcal{O}_2, \ldots$ of X there are finite $\mathcal{F}_1 \subseteq \mathcal{O}_1, \mathcal{F}_2 \subseteq \mathcal{O}_2, \ldots$ such that for each $x \in X$, the set $\{\, n \in \mathbb{N} : x \notin \bigcup \mathcal{F}_n \,\}$ is finite



The Hurewicz property

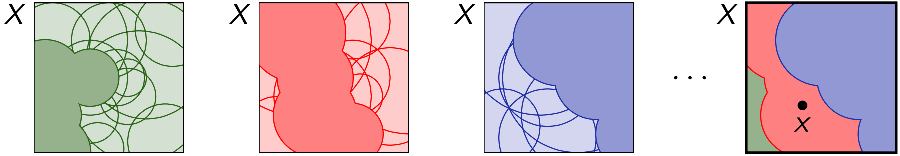
Hurewicz's property: for every sequence of open covers $\mathcal{O}_1, \mathcal{O}_2, \ldots$ of X there are finite $\mathcal{F}_1 \subseteq \mathcal{O}_1, \mathcal{F}_2 \subseteq \mathcal{O}_2, \ldots$ such that for each $x \in X$, the set $\{\, n \in \mathbb{N} : x \notin \bigcup \mathcal{F}_n \,\}$ is finite



$\mathcal{F}_1 \subseteq \mathcal{O}_1$ $\quad$ $\mathcal{F}_2 \subseteq \mathcal{O}_2$ $\quad$ $\mathcal{F}_3 \subseteq \mathcal{O}_3$ $\quad \ldots$

The Hurewicz property

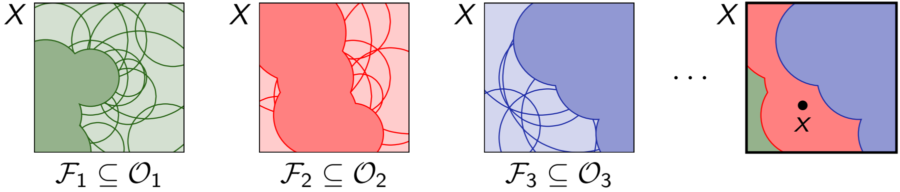
Hurewicz's property: for every sequence of open covers $\mathcal{O}_1, \mathcal{O}_2, \dots$ of X there are finite $\mathcal{F}_1 \subseteq \mathcal{O}_1, \mathcal{F}_2 \subseteq \mathcal{O}_2, \dots$ such that for each $x \in X$, the set $\{ n \in \mathbb{N} : x \notin \bigcup \mathcal{F}_n \}$ is finite



$\mathcal{F}_1 \subseteq \mathcal{O}_1$ $\mathcal{F}_2 \subseteq \mathcal{O}_2$ $\mathcal{F}_3 \subseteq \mathcal{O}_3$ $\dots$

The Hurewicz property

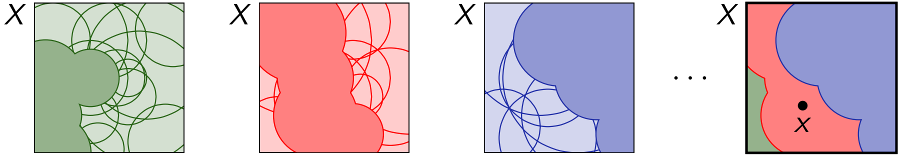
Hurewicz's property: for every sequence of open covers $\mathcal{O}_1, \mathcal{O}_2, \ldots$ of X there are finite $\mathcal{F}_1 \subseteq \mathcal{O}_1, \mathcal{F}_2 \subseteq \mathcal{O}_2, \ldots$ such that for each $x \in X$, the set $\{ n \in \mathbb{N} : x \notin \bigcup \mathcal{F}_n \}$ is finite



Hurewicz $\Rightarrow$ Menger

The Hurewicz property

Hurewicz's property: for every sequence of open covers $\mathcal{O}_1, \mathcal{O}_2, \ldots$ of X there are finite $\mathcal{F}_1 \subseteq \mathcal{O}_1, \mathcal{F}_2 \subseteq \mathcal{O}_2, \ldots$ such that for each $x \in X$, the set $\{\, n \in \mathbb{N} : x \notin \bigcup \mathcal{F}_n \,\}$ is finite

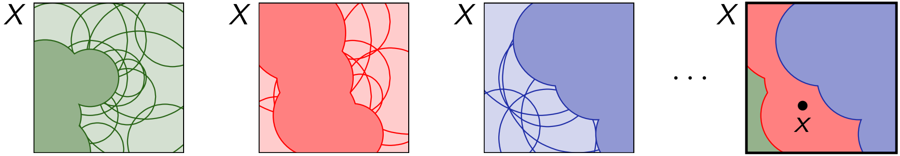


$\mathcal{F}_1 \subseteq \mathcal{O}_1$ $\qquad$ $\mathcal{F}_2 \subseteq \mathcal{O}_2$ $\qquad$ $\mathcal{F}_3 \subseteq \mathcal{O}_3$

σ-compact $\Rightarrow$ Hurewicz $\Rightarrow$ Menger

The Hurewicz property

Hurewicz's property: for every sequence of open covers $\mathcal{O}_1, \mathcal{O}_2, \ldots$ of X there are finite $\mathcal{F}_1 \subseteq \mathcal{O}_1, \mathcal{F}_2 \subseteq \mathcal{O}_2, \ldots$ such that for each $x \in X$, the set $\{\, n \in \mathbb{N} : x \notin \bigcup \mathcal{F}_n \,\}$ is finite



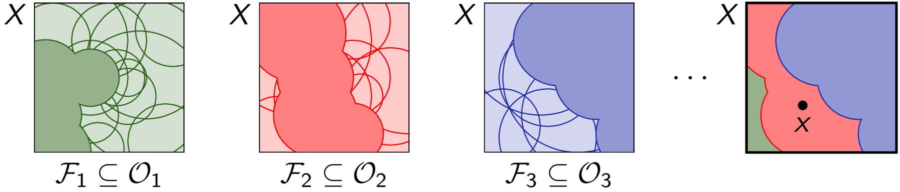
$\mathcal{F}_1 \subseteq \mathcal{O}_1$ $\mathcal{F}_2 \subseteq \mathcal{O}_2$ $\mathcal{F}_3 \subseteq \mathcal{O}_3$

σ-compact $\Rightarrow$ Hurewicz $\Rightarrow$ Menger

Aurichi, Tall ($\mathfrak{d} = \aleph_1$): metrizable productively Lindelöf $\Rightarrow$ Hurewicz

The Hurewicz property

Hurewicz's property: for every sequence of open covers $\mathcal{O}_1, \mathcal{O}_2, \ldots$ of X there are finite $\mathcal{F}_1 \subseteq \mathcal{O}_1, \mathcal{F}_2 \subseteq \mathcal{O}_2, \ldots$ such that for each $x \in X$, the set $\{\, n \in \mathbb{N} : x \notin \bigcup \mathcal{F}_n \,\}$ is finite



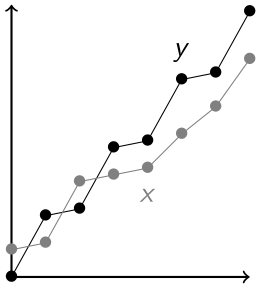
σ-compact $\Rightarrow$ Hurewicz $\Rightarrow$ Menger

Aurichi, Tall ($\mathfrak{d} = \aleph_1$): metrizable productively Lindelöf $\Rightarrow$ Hurewicz

Sz (ZFC): separable productively paracompact $\Rightarrow$ Hurewicz

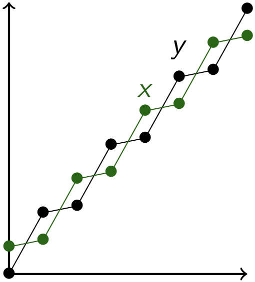
Hurewicz meets combinatorics

- $x \leq^* y$ if $x(n) \leq y(n)$ for almost all n



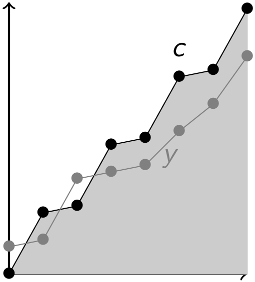
Hurewicz meets combinatorics

- $x \leq^* y$ if $x(n) \leq y(n)$ for almost all n
- $y \leq^\infty x$ if $x \not\leq^* y$



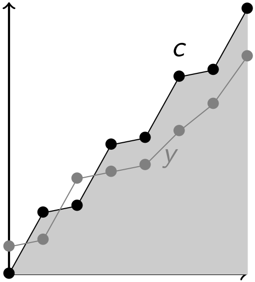
Hurewicz meets combinatorics

- $x \leq^* y$ if $x(n) \leq y(n)$ for almost all n
- $y \leq^\infty x$ if $x \not\leq^* y$
- Y is bounded if $\exists_{c \in [\mathbb{N}]^\infty} \forall_{y \in Y}\ y \leq^* c$



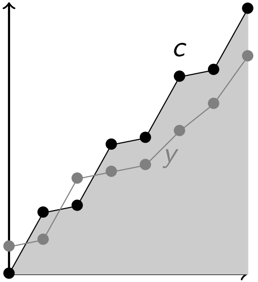
Hurewicz meets combinatorics

- $x \leq^* y$ if $x(n) \leq y(n)$ for almost all n
- $y \leq^\infty x$ if $x \not\leq^* y$
- Y is bounded if $\exists_{c \in [\mathbb{N}]^\infty} \forall_{y \in Y}\ y \leq^* c$
- $\mathfrak{b}$: minimal cardinality of an unbounded set



Hurewicz meets combinatorics

- $x \leq^* y$ if $x(n) \leq y(n)$ for almost all n
- $y \leq^\infty x$ if $x \not\leq^* y$
- Y is bounded if $\exists_{c \in [\mathbb{N}]^\infty} \forall_{y \in Y}\ y \leq^* c$
- $\mathfrak{b}$: minimal cardinality of an unbounded set



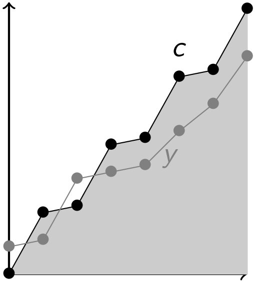
Theorem (Hurewicz)

Assume that X is Lindelöf and zero-dimensional

X is Hurewicz $\Leftrightarrow$ continuous image of X into $[\mathbb{N}]^\infty$ is **unbounded**

Hurewicz meets combinatorics

- $x \leq^* y$ if $x(n) \leq y(n)$ for almost all n
- $y \leq^\infty x$ if $x \not\leq^* y$
- Y is bounded if $\exists_{c \in [\mathbb{N}]^\infty} \forall_{y \in Y}\ y \leq^* c$
- $\mathfrak{b}$: minimal cardinality of an unbounded set

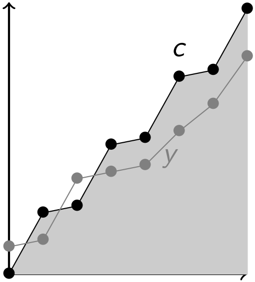


Theorem (Hurewicz)

Assume that X is Lindelöf and zero-dimensional
X is Hurewicz $\Leftrightarrow$ continuous image of X into $[\mathbb{N}]^\infty$ is **unbounded**

Hurewicz meets combinatorics

- $x \leq^* y$ if $x(n) \leq y(n)$ for almost all n
- $y \leq^\infty x$ if $x \not\leq^* y$
- Y is bounded if $\exists_{c \in [\mathbb{N}]^\infty} \forall_{y \in Y}\ y \leq^* c$
- $\mathfrak{b}$: minimal cardinality of an unbounded set



Theorem (Hurewicz)

Assume that X is Lindelöf and zero-dimensional
X is Hurewicz $\Leftrightarrow$ continuous image of X into $[\mathbb{N}]^\infty$ is **unbounded**

- A Lindelöf X with $|X| < \mathfrak{b}$ is Hurewicz
- An unbounded $X \subseteq [\mathbb{N}]^\infty$ is not Hurewicz

Main theorem again

$A \subseteq [\mathbb{N}]^\infty$ is $\mathfrak{d}$-unbounded if $|A| \geq \mathfrak{d}$ and $\forall_{\mathbf{c} \in [\mathbb{N}]^\infty} |\{ a \in A : a \leq \mathbf{c} \}| < \mathfrak{d}$

Theorem (Sz, Tsaban)

If $X \subseteq [\mathbb{N}]^\infty$ contains a $\mathfrak{d}$-unbounded set or a $\mathrm{cf}(\mathfrak{d})$-unbounded set, then there is a Menger $Y \subseteq \mathrm{P}(\mathbb{N})$, $X \times Y$ is not Menger

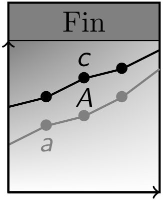
Main theorem again

$A \subseteq [\mathbb{N}]^\infty$ is $\mathfrak{d}$-unbounded if $|A| \geq \mathfrak{d}$ and $\forall_{\mathbf{c} \in [\mathbb{N}]^\infty} |\{ a \in A : a \leq \mathbf{c} \}| < \mathfrak{d}$

Theorem (Sz, Tsaban)

If $X \subseteq [\mathbb{N}]^\infty$ contains a $\mathfrak{d}$-unbounded set or a $\mathrm{cf}(\mathfrak{d})$-unbounded set, then there is a Menger $Y \subseteq \mathrm{P}(\mathbb{N})$, $X \times Y$ is not Menger

$Y = A \cup \mathrm{Fin}$, A is $\mathfrak{d}$-unbounded



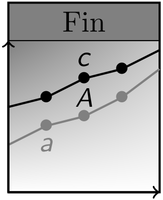
Main theorem again

$A \subseteq [\mathbb{N}]^\infty$ is $\mathfrak{d}$-unbounded if $|A| \geq \mathfrak{d}$ and $\forall_{\mathbf{c} \in [\mathbb{N}]^\infty} |\{ a \in A : a \leq \mathbf{c} \}| < \mathfrak{d}$

Theorem (Sz, Tsaban)

If $X \subseteq [\mathbb{N}]^\infty$ contains a $\mathfrak{d}$-unbounded set or a $\mathrm{cf}(\mathfrak{d})$-unbounded set, then there is a Menger $Y \subseteq \mathrm{P}(\mathbb{N})$, $X \times Y$ is not Menger

$Y = A \cup \mathrm{Fin}$, A is $\mathfrak{d}$-unbounded



Tsaban, Zdomskyy:

H is Hurewicz and hereditarily Lindelöf $\Rightarrow$ $H \times Y$ is Menger

Productivity of Menger and Hurewicz

X is productively Menger if for each Menger M, $X \times M$ is Menger

Productivity of Menger and Hurewicz

X is productively Menger if for each Menger M, $X \times M$ is Menger

Theorem (Sz, Tsaban) $\mathfrak{b} = \mathfrak{d}$, hereditarily Lindelöf spaces

productively Menger $\Rightarrow$ productively Hurewicz

Productivity of Menger and Hurewicz

X is productively Menger if for each Menger M, $X \times M$ is Menger

Theorem (Sz, Tsaban) $\mathfrak{b} = \mathfrak{d}$, hereditarily Lindelöf spaces

productively Menger $\Rightarrow$ productively Hurewicz

- Asm X prod Menger, $X \times H$ not Hurewicz

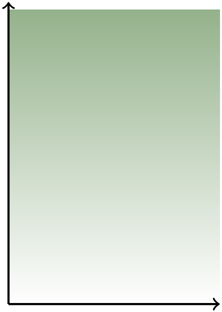
Productivity of Menger and Hurewicz

X is productively Menger if for each Menger M, $X \times M$ is Menger

Theorem (Sz, Tsaban) $\mathfrak{b} = \mathfrak{d}$, hereditarily Lindelöf spaces

productively Menger $\Rightarrow$ productively Hurewicz

- Asm X prod Menger, $X \times H$ not Hurewicz
- $X \times H \to Y \subseteq [\mathbb{N}]^\infty$ unbounded



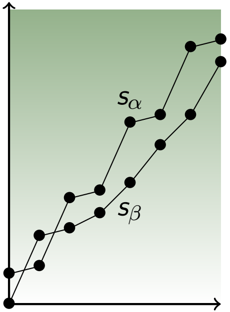
Productivity of Menger and Hurewicz

X is productively Menger if for each Menger M, $X \times M$ is Menger

Theorem (Sz, Tsaban) $\mathfrak{b} = \mathfrak{d}$, hereditarily Lindelöf spaces

productively Menger $\Rightarrow$ productively Hurewicz

- Asm X prod Menger, $X \times H$ not Hurewicz
- $X \times H \to Y \subseteq [\mathbb{N}]^\infty$ unbounded
- $(\mathfrak{b} = \mathfrak{d})$
  $\exists$ dominating $\{ s_\alpha : \alpha < \mathfrak{b} \}$, $s_\beta \leq^* s_\alpha$, $\beta \leq \alpha$



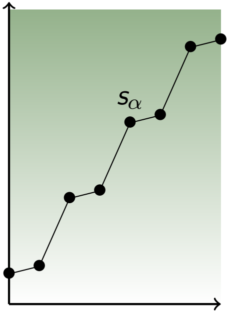
Productivity of Menger and Hurewicz

X is productively Menger if for each Menger M, $X \times M$ is Menger

Theorem (Sz, Tsaban) $\mathfrak{b} = \mathfrak{d}$, hereditarily Lindelöf spaces

productively Menger $\Rightarrow$ productively Hurewicz

- Asm X prod Menger, $X \times H$ not Hurewicz
- $X \times H \to Y \subseteq [\mathbb{N}]^\infty$ unbounded
- $(\mathfrak{b} = \mathfrak{d})$
  $\exists$ dominating $\{ s_\alpha : \alpha < \mathfrak{b} \}$, $s_\beta \leq^* s_\alpha$, $\beta \leq \alpha$
- $s_\alpha \leq^\infty y_\alpha \in Y$



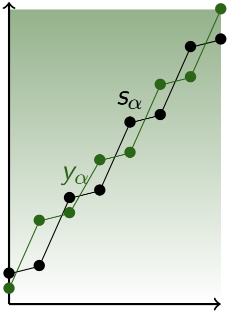
Productivity of Menger and Hurewicz

X is productively Menger if for each Menger M, $X \times M$ is Menger

Theorem (Sz, Tsaban) $\mathfrak{b} = \mathfrak{d}$, hereditarily Lindelöf spaces

productively Menger $\Rightarrow$ productively Hurewicz

- Asm X prod Menger, $X \times H$ not Hurewicz
- $X \times H \to Y \subseteq [\mathbb{N}]^\infty$ unbounded
- $(\mathfrak{b} = \mathfrak{d})$
  $\exists$ dominating $\{ s_\alpha : \alpha < \mathfrak{b} \}$, $s_\beta \leq^* s_\alpha$, $\beta \leq \alpha$
- $s_\alpha \leq^\infty y_\alpha \in Y$



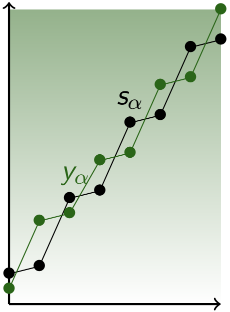
Productivity of Menger and Hurewicz

X is productively Menger if for each Menger M, $X \times M$ is Menger

Theorem (Sz, Tsaban) $\mathfrak{b} = \mathfrak{d}$, hereditarily Lindelöf spaces

productively Menger $\Rightarrow$ productively Hurewicz

- Asm X prod Menger, $X \times H$ not Hurewicz
- $X \times H \to Y \subseteq [\mathbb{N}]^\infty$ unbounded
- $(\mathfrak{b} = \mathfrak{d})$
  $\exists$ dominating $\{ s_\alpha : \alpha < \mathfrak{b} \}$, $s_\beta \leq^* s_\alpha$, $\beta \leq \alpha$
- $s_\alpha \leq^\infty y_\alpha \in Y$
- $\mathfrak{d}$-unbounded $\{ y_\alpha : \alpha < \mathfrak{b} \} \subseteq Y$



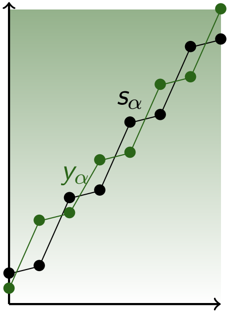
Productivity of Menger and Hurewicz

X is productively Menger if for each Menger M, $X \times M$ is Menger

Theorem (Sz, Tsaban) $\mathfrak{b} = \mathfrak{d}$, hereditarily Lindelöf spaces

productively Menger $\Rightarrow$ productively Hurewicz

- Asm X prod Menger, $X \times H$ not Hurewicz
- $X \times H \to Y \subseteq [\mathbb{N}]^\infty$ unbounded
- $(\mathfrak{b} = \mathfrak{d})$
  $\exists$ dominating $\{ s_\alpha : \alpha < \mathfrak{b} \}$, $s_\beta \leq^* s_\alpha$, $\beta \leq \alpha$
- $s_\alpha \leq^\infty y_\alpha \in Y$
- $\mathfrak{d}$-unbounded $\{ y_\alpha : \alpha < \mathfrak{b} \} \subseteq Y$
- $\exists$ Menger $M \subseteq \mathrm{P}(\mathbb{N})$, $Y \times M$ not Menger



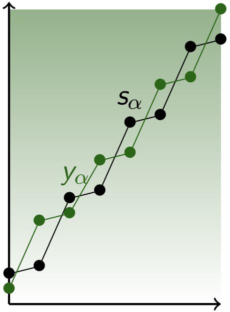
Productivity of Menger and Hurewicz

X is productively Menger if for each Menger M, $X \times M$ is Menger

Theorem (Sz, Tsaban) $\mathfrak{b} = \mathfrak{d}$, hereditarily Lindelöf spaces

productively Menger $\Rightarrow$ productively Hurewicz

- Asm X prod Menger, $X \times H$ not Hurewicz
- $X \times H \to Y \subseteq [\mathbb{N}]^\infty$ unbounded
- $(\mathfrak{b} = \mathfrak{d})$
  $\exists$ dominating $\{ s_\alpha : \alpha < \mathfrak{b} \}$, $s_\beta \leq^* s_\alpha$, $\beta \leq \alpha$
- $s_\alpha \leq^\infty y_\alpha \in Y$
- $\mathfrak{d}$-unbounded $\{ y_\alpha : \alpha < \mathfrak{b} \} \subseteq Y$
- $\exists$ Menger $M \subseteq \mathrm{P}(\mathbb{N})$, $Y \times M$ not Menger
- $(X \times H) \times M \to Y \times M$, $(X \times H) \times M$ not Menger



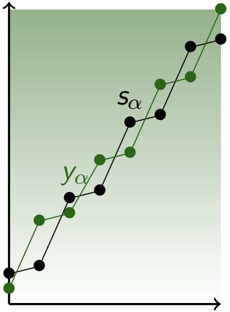
Productivity of Menger and Hurewicz

X is productively Menger if for each Menger M, $X \times M$ is Menger

Theorem (Sz, Tsaban) $\mathfrak{b} = \mathfrak{d}$, hereditarily Lindelöf spaces

productively Menger $\Rightarrow$ productively Hurewicz

- Asm X prod Menger, $X \times H$ not Hurewicz
- $X \times H \to Y \subseteq [\mathbb{N}]^\infty$ unbounded
- $(\mathfrak{b} = \mathfrak{d})$
  $\exists$ dominating $\{ s_\alpha : \alpha < \mathfrak{b} \}$, $s_\beta \leq^* s_\alpha$, $\beta \leq \alpha$
- $s_\alpha \leq^\infty y_\alpha \in Y$
- $\mathfrak{d}$-unbounded $\{ y_\alpha : \alpha < \mathfrak{b} \} \subseteq Y$
- $\exists$ Menger $M \subseteq \mathrm{P}(\mathbb{N})$, $Y \times M$ not Menger
- $(X \times H) \times M \to Y \times M$, $(X \times H) \times M$ not Menger
- $H \times M$ is Menger, $X \times (H \times M)$ is Menger



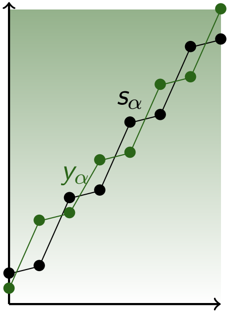
Productivity of Menger and Hurewicz

X is productively Menger if for each Menger M, $X \times M$ is Menger

Theorem (Sz, Tsaban) $\mathfrak{b} = \mathfrak{d}$, hereditarily Lindelöf spaces

productively Menger $\Rightarrow$ productively Hurewicz

- Asm X prod Menger, $X \times H$ not Hurewicz
- $X \times H \to Y \subseteq [\mathbb{N}]^\infty$ unbounded
- $(\mathfrak{b} = \mathfrak{d})$
  $\exists$ dominating $\{ s_\alpha : \alpha < \mathfrak{b} \}$, $s_\beta \leq^* s_\alpha$, $\beta \leq \alpha$
- $s_\alpha \leq^\infty y_\alpha \in Y$
- $\mathfrak{d}$-unbounded $\{ y_\alpha : \alpha < \mathfrak{b} \} \subseteq Y$
- $\exists$ Menger $M \subseteq \mathrm{P}(\mathbb{N})$, $Y \times M$ not Menger
- $(X \times H) \times M \to Y \times M$, $(X \times H) \times M$ not Menger
- $H \times M$ is Menger, $X \times (H \times M)$ is Menger ⚡⚡⚡



Productivity of Menger and Hurewicz

X is productively Menger if for each Menger M, $X \times M$ is Menger

Theorem (Sz, Tsaban) $\mathfrak{b} = \mathfrak{d}$, hereditarily Lindelöf spaces

productively Menger $\Rightarrow$ productively Hurewicz

What about general spaces?

Productivity of Menger and Hurewicz

X is productively Menger if for each Menger M, $X \times M$ is Menger

Theorem (Sz, Tsaban) $\mathfrak{b} = \mathfrak{d}$, hereditarily Lindelöf spaces

productively Menger $\Rightarrow$ productively Hurewicz

What about general spaces?

Miller, Tsaban, Zdomskyy ($\mathfrak{d} = \aleph_1$):
metrizable productively Lindelöf $\Rightarrow$ productively Hurewicz
metrizable productively Lindelöf $\Rightarrow$ productively Menger

Productivity of Menger and Hurewicz

X is productively Menger if for each Menger M, $X \times M$ is Menger

Theorem (Sz, Tsaban) $\mathfrak{b} = \mathfrak{d}$, hereditarily Lindelöf spaces

productively Menger $\Rightarrow$ productively Hurewicz

What about general spaces?

Miller, Tsaban, Zdomskyy ($\mathfrak{d} = \aleph_1$):
metrizable productively Lindelöf $\Rightarrow$ productively Hurewicz
metrizable productively Lindelöf $\Rightarrow$ productively Menger

Theorem (Sz, Tsaban) $\mathfrak{d} = \aleph_1$, general spaces

productively Lindelöf $\Rightarrow$ productively Menger $\Rightarrow$ productively Hurewicz

Open problems

Sets of reals

Open problems

Sets of reals

- $\mathfrak{d} \leq \mathfrak{r} \Leftrightarrow$ Menger is not productive?

Open problems

Sets of reals

- $\mathfrak{d} \leq \mathfrak{r} \Leftrightarrow$ Menger is not productive?
- Any Lusin set is not productively Menger?

Open problems

Sets of reals

- $\mathfrak{d} \leq \mathfrak{r} \Leftrightarrow$ Menger is not productive?
- Any Lusin set is not productively Menger?

Hereditarily Lindelöf spaces

Open problems

Sets of reals

- $\mathfrak{d} \leq \mathfrak{r} \Leftrightarrow$ Menger is not productive?
- Any Lusin set is not productively Menger?

Hereditarily Lindelöf spaces

- $(\mathfrak{b} = \mathfrak{d})$ productively Menger $\neq$ productively Hurewicz?

Open problems

Sets of reals

- $\mathfrak{d} \leq \mathfrak{r} \Leftrightarrow$ Menger is not productive?
- Any Lusin set is not productively Menger?

Hereditarily Lindelöf spaces

- $(\mathfrak{b} = \mathfrak{d})$ productively Menger $\neq$ productively Hurewicz?

General spaces

Open problems

Sets of reals

- $\mathfrak{d} \leq \mathfrak{r} \Leftrightarrow$ Menger is not productive?
- Any Lusin set is not productively Menger?

Hereditarily Lindelöf spaces

- $(\mathfrak{b} = \mathfrak{d})$ productively Menger $\neq$ productively Hurewicz?

General spaces

- $X \subseteq \mathbb{R}$, $|X| < \mathfrak{b} \Rightarrow X \times$ Hurewicz is Lindelöf?

Open problems

Sets of reals

- $\mathfrak{d} \leq \mathfrak{r} \Leftrightarrow$ Menger is not productive?
- Any Lusin set is not productively Menger?

Hereditarily Lindelöf spaces

- $(\mathfrak{b} = \mathfrak{d})$ productively Menger $\neq$ productively Hurewicz?

General spaces

- $X \subseteq \mathbb{R}$, $|X| < \mathfrak{b} \Rightarrow X\times$ Hurewicz is Lindelöf?
- $(\mathfrak{d} = \aleph_1)$ productively Menger $\neq$ productively Hurewicz

Open problems

Sets of reals

- $\mathfrak{d} \leq \mathfrak{r} \Leftrightarrow$ Menger is not productive?
- Any Lusin set is not productively Menger?

Hereditarily Lindelöf spaces

- $(\mathfrak{b} = \mathfrak{d})$ productively Menger $\neq$ productively Hurewicz?

General spaces

- $X \subseteq \mathbb{R}$, $|X| < \mathfrak{b} \Rightarrow X \times$ Hurewicz is Lindelöf?
- $(\mathfrak{d} = \aleph_1)$ productively Menger $\neq$ productively Hurewicz
- Any Sierpiński set is not productively Hurewicz? is not productively Menger? under CH?