On hyperfiniteness of boundary actions of hyperbolic groups

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This is joint work (in progress) with Jingyin Huang and Forte Shinko.

Definition (δ-hyperbolic space)

Suppose X is a geodesic metric space, $\delta > 0$ and $x, y, z \in X$. A geodesic triangle whose sides are geodesic segments $[x, y]$, $[y, z]$ and $[z, x]$ is called δ-*slim* if any of the three above geodesic segments is in the δ-neighborhood of the two remaining sides.

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In general, the smaller δ is, the more δ-hyperbolic spaces "look like" trees.

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Examples

There are many examples of hyperbolic groups. The free groups F_n are of course hyperbolic. All fundamental groups $\pi_1(M)$ of compact hyperbolic manifolds M are hyperbolic.

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Given a hyperbolic space X with a distinguished point O we identify two geodesic rays γ_1 and γ_2 (write $\gamma_1 \sim \gamma_2$) if there exists a constant $K > 0$ such that

$$d(\gamma_1(t), \gamma_2(t)) < K$$

for all t. The *boundary of* X, denoted ∂X is the set of all $\sim$-classes of geodesic rays in X.

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Thus defined, ∂X is just a set and it carries a natural compact topology.

Definition (Gromov product)

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Topology on the boundary

Given $p \in \partial X$ and $r > 0$ we define the neighborhood of p as

$$\{q \in \partial X : \exists \gamma \in q, \exists \gamma' \in p \quad \inf_{s,t \to \infty} (\gamma(s), \gamma'(t))_O \geq r\}$$

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With the above topology, the boundary is a compact topological metrizable space.

Boundary of a hyperbolic group

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Suppose Γ is a hyperbolic group and $p \in \partial\Gamma$. Let $\gamma \in p$ be a geodesic ray. For any $g \in \Gamma$ there exists a unique geodesic ray starting at e which hits the geodesic $\gamma'(t) = g \cdot \gamma(t)$. Denote this geodesic ray by $g\gamma$.

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Boundary action

The above $(g, p) \mapsto [g\gamma]_\sim$ induces an action of Γ the boundary $\partial\Gamma$ which is called the *boundary action*.

Alternate definition

An alternate definition of the boundary is to take all possible infinite geodesics (not neccessarily starting at e) modded out by finite Hausdorff distance. In this definition the boundary action is more natural.

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Anyhow, the action of Γ on its boundary is an action by homeomorphisms.

Definition

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Sometimes, Borel equivalence relations arise from Borel actions of countable groups $\Gamma \curvearrowright X$.

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In general, the group, although countable, may, however be quite complicated.

(post-)Definition

A countable equivalence relation is called *hyperfinite* if it induced by a Borel action of $\mathbb{Z}$.

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Theorem (Slaman–Steel, Weiss)

For a Borel countable equivalence relation E, the following are equivalent:

- E is hyperfinite,
- E is an increasing union of Borel equivalence relations E_n such that each E_n has finite classes.

Definition

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Theorem (Adams)

Suppose Γ is a hyperbolic group. If μ is any Borel probability measure on the boundary $\partial\Gamma$, then the action of Γ on $\partial\Gamma$ is μ-a.e. hyperfinite.

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Hyperfinite vs μ-a.e. hyperfinite

The distinction between equivalence relations which are μ-a.e. hyperfinite from those which are hyperfinite is a well-known (and very hard) problem in measurable dynamics and motivates the following question.

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Is the boundary action of every hyperbolic group hyperfinite?

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We can provide positive answer for a large class of hyperbolic groups.

Definition

The *tail equivalence relation* E_t is the equivalence relation defined at $2^{\mathbb{N}}$ as follows:

$$x \; E_t \; y \quad \text{if} \quad \exists n, m \; \forall k \quad x(n+k) = y(m+k)$$

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Remark

It is not difficult to see that the tail equivalence relation is Borel-bireducible with the action of the free group F_2 on its boundary Cantor set.

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Theorem (Dougherty–Jackson–Kechris)

The tail equivalence relation is hyperfinite.

Model triangles

Suppose X is a geodesic metric space. Given three points $x, y, z \in X$ and geodesic segments $[x, y], [y, z], [z, x]$ consider a corresponding triangle x', y', z' on the Euclidean plane with the lengths of $[x', y'], [y', z'], [z', x']$ equal to the corresponding lengths $[x, y], [y, z], [z, x]$.

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For any two points $p, q \in [x, y] \cup [y, z] \cup [z, x]$ there exist unique $p', q' \in\in [x, y] \cup [y, z] \cup [z, x]$ which divide the sides in the same proportion.

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Definition

The space X is *CAT(0)* if the for any $x, y, z, p, q \in X$ as above we have $d(p, q) \leq d_e(p', q')$ where d_e is the Euclidean distance.

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Link of a vertex

Given a vertex v in a cube complex X, the link of v is the complex built of simplices whose vertices correspond to the edges of X whose one of the endpoints is v. Simplices are spanned by those collections of edges which are corners of cubes.

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Alternately, the link of a vertex v can be seen as the intersection of the complex with a small sphere around the vertex v.

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Fact

A cube complex is CAT(0) if and only if it is simply connected and the link of every vertex is a flag complex.

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The action of a group on a cube complex is *cocompact* if there are finitely many orbits.

It turns out that if a group acts properly and cocompactly on a complex, then one can deduce many properties of the group from the properties of the complex.

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Hyperbolic groups

For example, such a group is hyperbolic if and only if the complex is hyperbolic.

It turns out that many hyperbolic groups act properly and cocompactly on CAT(0) cube complexes

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Theorem (Bergeron–Wise, Kahn–Markovic)

All fundamental groups of hyperbolic closed 3-manifolds admit proper cocompact actions on CAT(0) cube complexes.

Theorem (Huang–S.–Shinko)

If a hyperbolic group Γ acts properly and cocompactly on a CAT(0) cube complex, then the boundary action of Γ on $\partial\Gamma$ is hyperfinite.

Boundary of a complex

Given a proper and cocompact action of a hyperbolic group Γ on a complex X one can define the boundary of this action in a similar way as the boundary of the group.

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Hyperfiniteness

If a hyperbolic group acts properly and cocompactly on a complex, then the this induces an action on the boundary of the complex, which is hyperfinite if and only if the boundary action of the group is hyperfinite.

Theorem (Huang–S.–Shinko)

If a hyperbolic group Γ acts properly and cocompactly on a CAT(0) cube complex X, then the induced action $\Gamma \curvearrowright \partial X$ is hyperfinite.