Equivariant geometry of Banach spaces and topological groups

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An analogous concept due to M. Gromov is also available for preservation of the large scale geometry.

Namely, a map $\phi\colon X \to Y$ is a coarse embedding if, for all sequences x_n, z_n in X,

$$d(x_n, z_n) \underset{n\to\infty}{\longrightarrow} \infty \quad \Leftrightarrow \quad d\big(\phi(x_n), \phi(z_n)\big) \underset{n\to\infty}{\longrightarrow} \infty.$$

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This is known, for example, for $Y = \mathcal{H}$ Hilbert space by a result of N. L. Randrianarivony.

Observe that, if $\phi\colon X \to Y$ is either a uniform or a coarse embedding, then there are $\Delta, \delta > 0$ so that

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E.g., if X uniformly embeds into ℓ^p, then X coarsely embeds into $\ell^p = \ell^p(\ell^p)$.

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There is a bornologous map $\phi\colon X \to Y$ between separable Banach spaces which isn't close to any uniformly continuous map $\psi\colon X \to Y$.

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Also, bornologous is a large scale property enjoyed by every uniformly continuous map.

J. Roe's Coarse spaces

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A coarse space is a set X equipped with an ideal $\mathcal{E}$ of subsets $E \subseteq X \times X$ so that $\Delta_X \in \mathcal{E}$ and

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For example, if (X, d) is a metric space, its corresponding coarse structure $\mathcal{E}_d$ is the ideal generated by sets of the form

$$E_\alpha = \{(x, y) \in X \times X \mid d(x, y) < \alpha\}$$

where $\alpha < \infty$.

The left-invariant coarse structure of a topological group

Theorem (G. Birkhoff – S. Kakutani – A. Weil)

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and the intersection is taken over all left-invariant continuous pseudo-metrics d *on* G.

Examples

The coarse structure $\mathcal{E}_L$ on a finitely generated discrete group Γ is that induced by the word metric ρ_S of any finite generating set $S \subseteq \Gamma$.

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For many other groups, the coarse structure may be computed explicitly.

Henceforth, we only consider topological groups whose coarse structure $\mathcal{E}_L$ is induced by a single left-invariant compatible metric d, i.e.,

$$\mathcal{E}_L = \mathcal{E}_d.$$

Linear and affine representations

Let E be a Banach space and G a topological group. A continuous isometric linear representation of G on E is a continuous action

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into the group $\mathrm{Isom}(E)$ of linear isometries of E, equipped with the strong operator topology, that is, the topology of pointwise convergence on E.

By a result of Mazur and Ulam, every surjective isometry $A\colon E \to E$ of a Banach space is affine, that is, of the form

$$A(\xi) = T(\xi) + \eta_0$$

for some linear isometry T and vector $\eta_0 \in E$.

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I.e., for $g \in G$ and $\xi \in E$,

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A coarsely proper continuous affine isometric action $\alpha\colon G \curvearrowright E$ may be viewed as an action that faithfully represents the coarse geometry of G.

The Haagerup property

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In the context of countable or locally compact groups, the Haagerup property is often viewed as a strong non-rigidity property.

For general Polish groups, we may also view it as a regularity property, since it allows for an efficient representation of G on the most regular Banach space $\mathcal{H}$.

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A geometric particuliarity of $\mathcal{H}$ used here is that a metric space coarsely embeds into $\mathcal{H}$ if and only if it has a uniformly continuous coarse embedding into $\mathcal{H}$.

Local properties

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For example super-reflexivity and super-stability.

Another take on amenability

A locally compact group G is amenable if and only if it admits a Følner sequence, that is, a sequence $F_1, F_2, \ldots \subseteq G$ of compact sets so that

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E.g., the unitary subgroup $U(M)$ of an approximately finite-dimensional von Neumann algebra M is approximately compact (P. de la Harpe).

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For example, every abelian Polish group is Følner amenable. E.g., Banach spaces.

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E.g., the property of being super-reflexive (Clarkson), that is, having a uniformly convex renorming (Enflo).

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Coupling a quantitative version of the above result with work of Krivine–Maurey and Raynaud, we obtain the following.

Corollary

Let X be a Banach space uniformly embeddable into the unit ball B_E of a super-reflexive Banach space E. Then X contains an isomorphic copy of some ℓ^p, $1 \leqslant p < \infty$.

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Problem

Is every Polish amenable group also Følner amenable?

To our knowledge, this is still open, though a simple counter-example may exist.

Polish groups of bounded geometry

Definition

A Polish group has bounded geometry if it is coarsely equivalent to a proper metric space.

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$$\sup_{y \in Y} d(y, \phi[X]) < \infty.$$

Since the coarse structure of a locally compact second countable group is given by a proper metric on the group, every such group has bounded geometry.

Consider the central extension

$$\mathbb{Z} \to \mathrm{Homeo}_{\mathbb{Z}}(\mathbb{R}) \to \mathrm{Homeo}_{+}(\mathbb{S}^1),$$

where $\mathrm{Homeo}_{\mathbb{Z}}(\mathbb{R})$ is the group of homeomorphisms of $\mathbb{R}$ commuting with integral shifts.

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Corollary

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Also, for Følner amenable groups with faithful unitary representations, we may have very strong geometric obstructions.

Theorem

Every continuous affine isometric action of $\mathrm{Isom}(\mathbb{Z}\mathbb{U})$ *on a reflexive Banach space or on* $L^1([0,1])$ *has a fixed point.*

However, combining amenability and bounded geometry, we obtain an analogue of the Brown–Guentner Theorem.

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The main idea here is to produce a sequence $\phi_n \colon G \to \ell^{p_n}$ of uniformly continuous maps that sufficiently separate points of G.

Using amenability, each of the ϕ_n are averaged to produce cocycles $b_n \colon G \to L^{p_n}$, so that the cocycle

$$b = b_1 \oplus b_2 \oplus \ldots$$

with values in the reflexive space $\bigoplus_n L^{p_n}$ is coarsely proper.

Reflexive spaces

Theorem

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Here d is stable if, for all bounded sequences (x_n) and (y_m) and all ultrafilters $\mathcal{U}$ and $\mathcal{V}$, we have

$$\lim_{n\to\mathcal{U}} \lim_{m\to\mathcal{V}} d(x_n, y_m) = \lim_{m\to\mathcal{V}} \lim_{n\to\mathcal{U}} d(x_n, y_m).$$

In the context of automorphism groups of countable first-order structures, we have the following corollary.

Corollary

Let $\mathbf{A}$ *be a countable atomic model of a stable theory* T *and assume that* $\mathrm{Aut}(\mathbf{A})$ *has metrisable coarse structure.*
Then $\mathrm{Aut}(\mathbf{A})$ *admits a coarsely proper continuous affine isometric action on a reflexive Banach space.*

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Then $\mathrm{Aut}(\mathbf{A})$ *admits a coarsely proper continuous affine isometric action on a reflexive Banach space.*

By a result of J. Zielinski, the assumption of metrisability is not automatic from the other hypotheses.