History, Structure, Results and Problems on Hyperspaces and Symmetric Products

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Introduction

- In general topology, given a space X there are several ways to construct a new space **K(X)** from X.

A **continuum** is a compact connected metric space

$$2^X = \{ A \subset X : A \text{ is closed and } A \neq \emptyset \},$$

$$C(X) = \{ A \in 2^X : A \text{ is connected} \},$$

$$C_n(X) = \{ A \in 2^X : A \text{ has at most n components} \}$$

$$F_n(X) = \{ A \in 2^X : A \text{ has at most n points} \},$$

$$F_1(X) = \{ \{p\} : p \in X \}.$$

$C(X)$, $Cn(X)$ and $F_1(X)$

- Note that $C(X)=C_1(X)$ and $F_1(X)$ is homeomorphic to X.
- The hyperspaces $C(X)$, $C_n(X)$ and $F_n(X)$ are considered with the ***Vietoris Topology*** (Hausdorff Metric)

Hyperspaces and Symmetric Products

- $C(X)$=***hyperspace of subcontinua***
- $C_n(X)$= ***n-fold hyperspace***
- $F_n(X)$=***n^{th}-symmetric product***

Hyperspaces and Symmetric Products

- Given a hyperspace

  $K(X) \in \{2^X, Cn\ (X), Fn(X)\}$

  there are several natural problems in the sructure of Hyperspaces.
- We discuss three in this talk:

$K(X) \in \{2^X, Cn(X), Fn(X)\}$

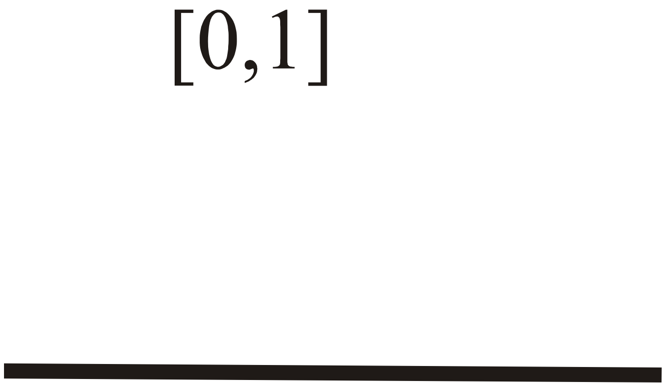
- (I) For which continua X is the hyperspace K(X) a cone.
- (II) When does X have unique hyperspace K(X)?
- (III) Determine the homogeneity degree of a hyperspace K(X).

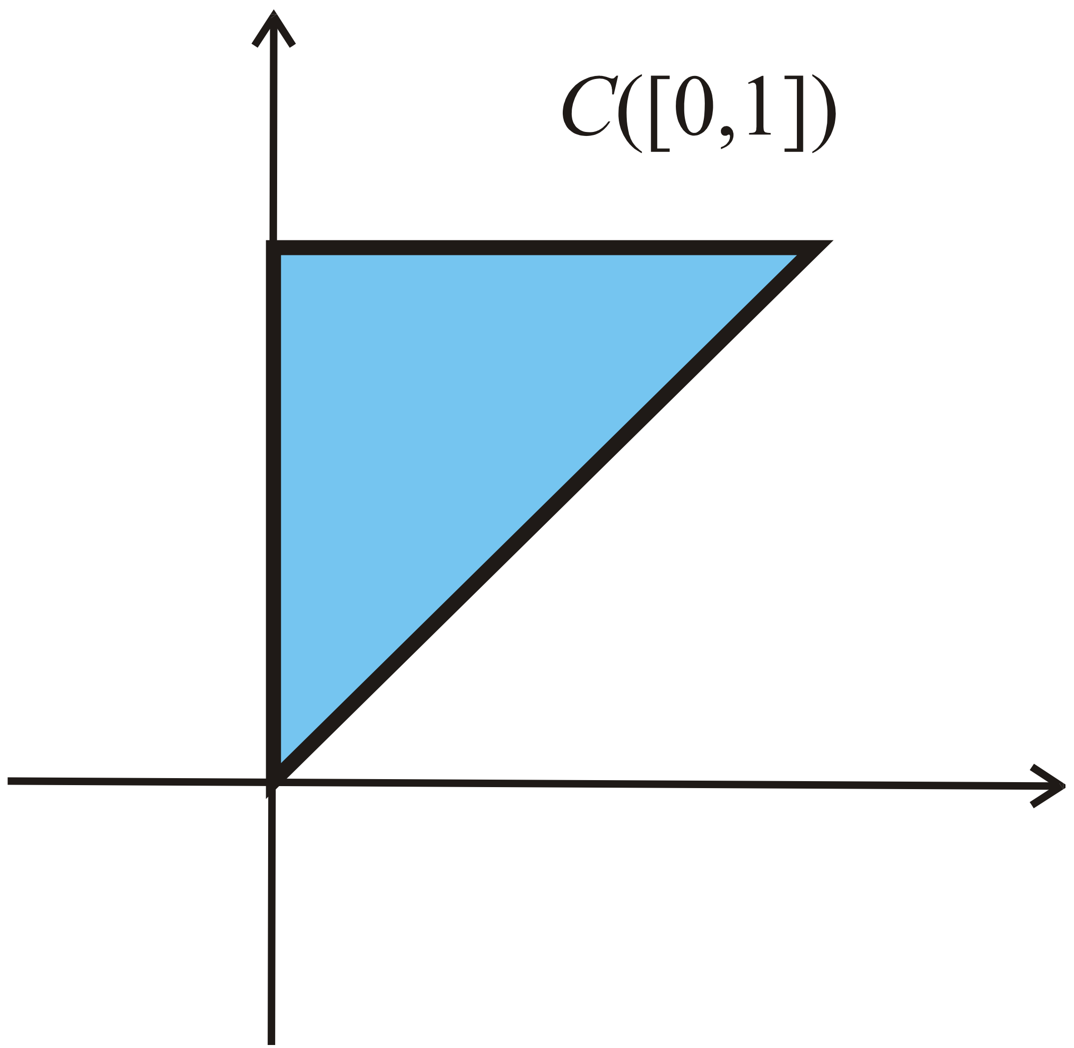
PROBLEM I
HYPERSPACES AND CONES

Hyperspaces and Cones

- A continumm X is a ***cone*** provided that there exists a space **Z** such that X is homeomorphic to the cone of Z.

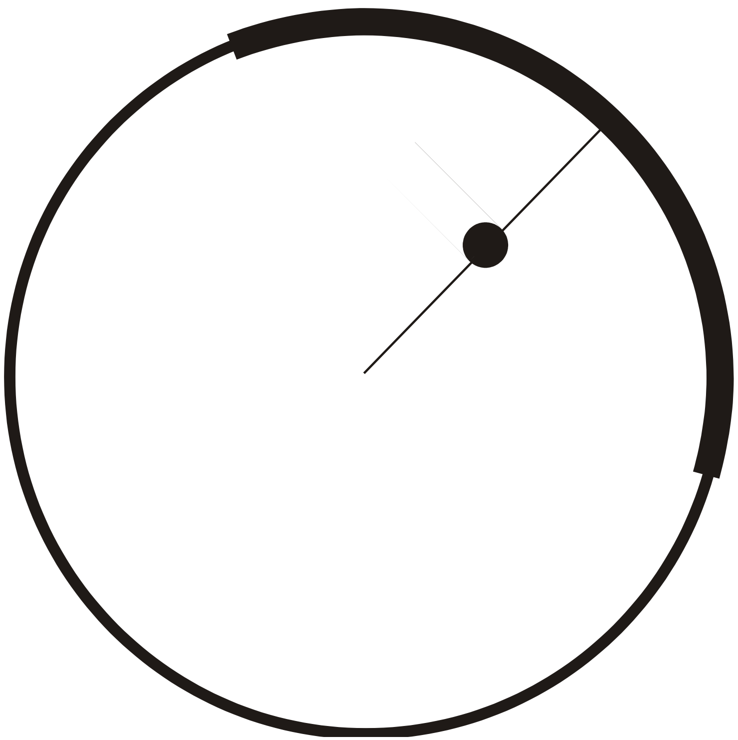
$$C([0,1]) = \{ [a,b] : 0 \le a \le b \le 1 \}$$
$$\sim \{ (a,b) \in R^2 : 0 \le a \le b \le 1 \}.$$

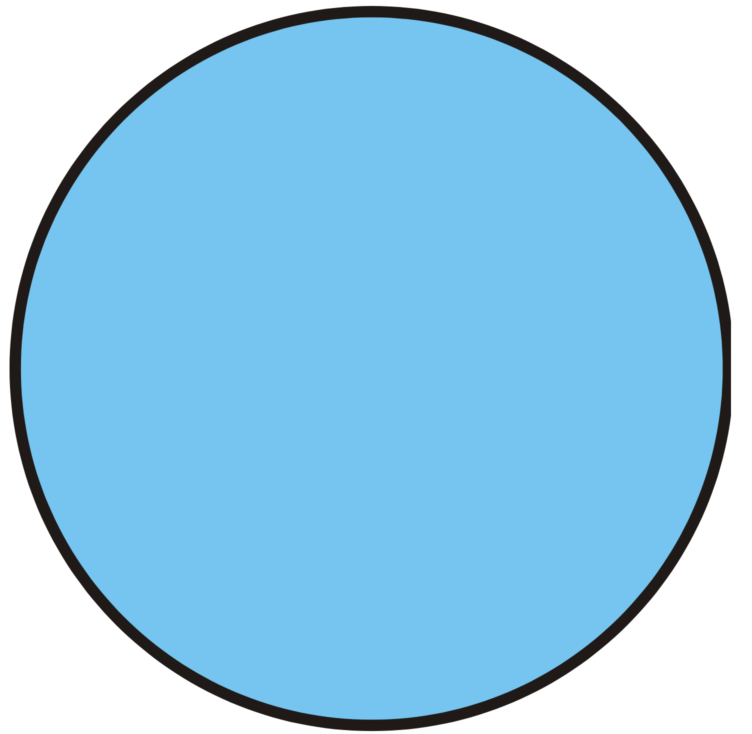




$C(S^1)$

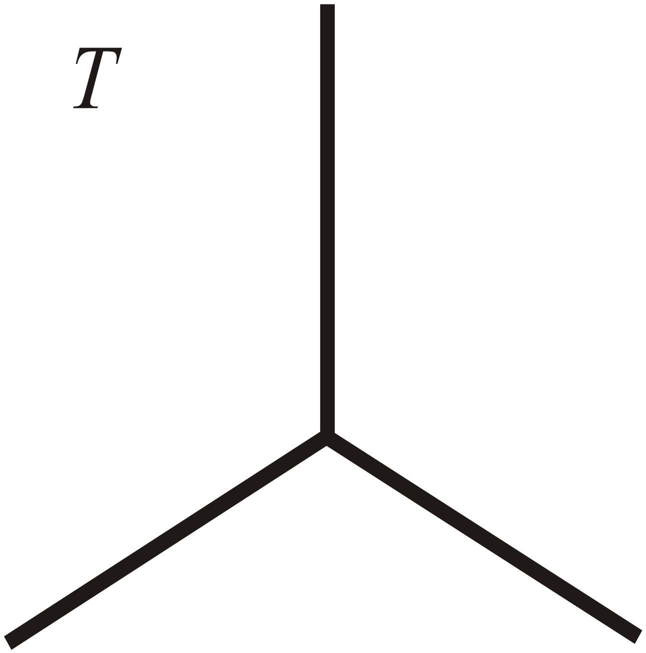
$$q(A) = (1 - l(A)/2\pi)\,m(A)$$



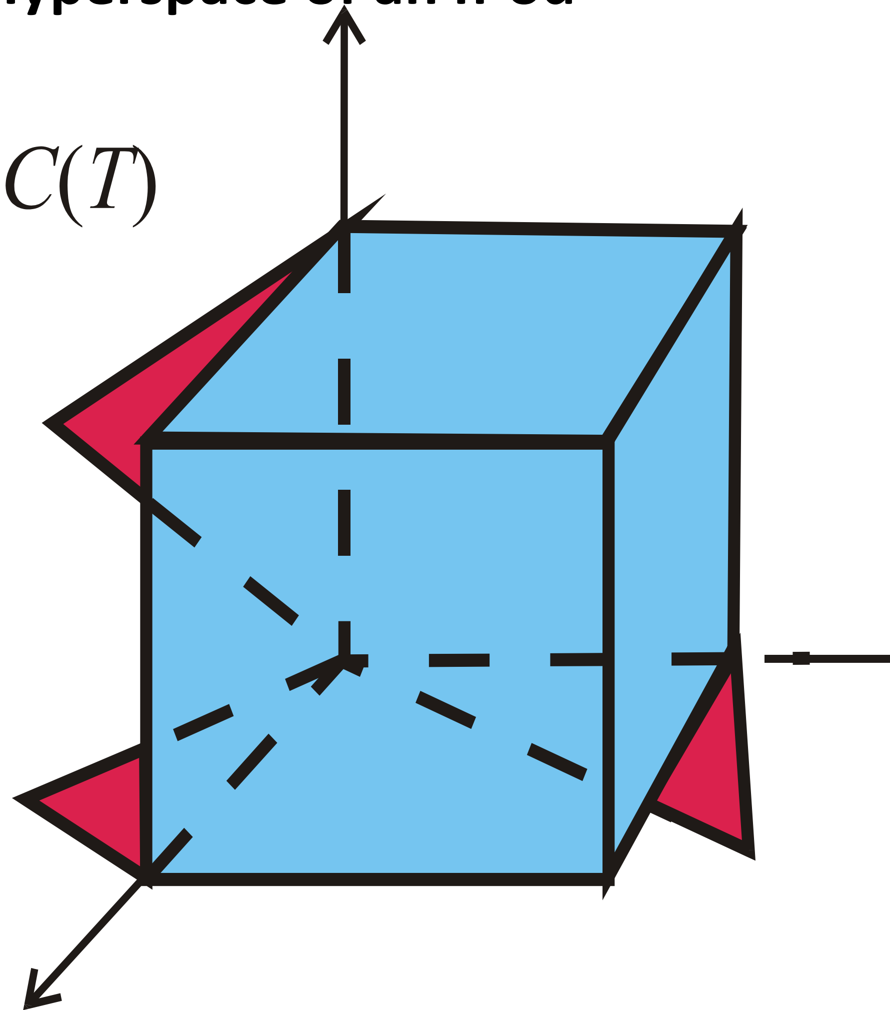


C(T)

n-od



Hyperspace of an n-od



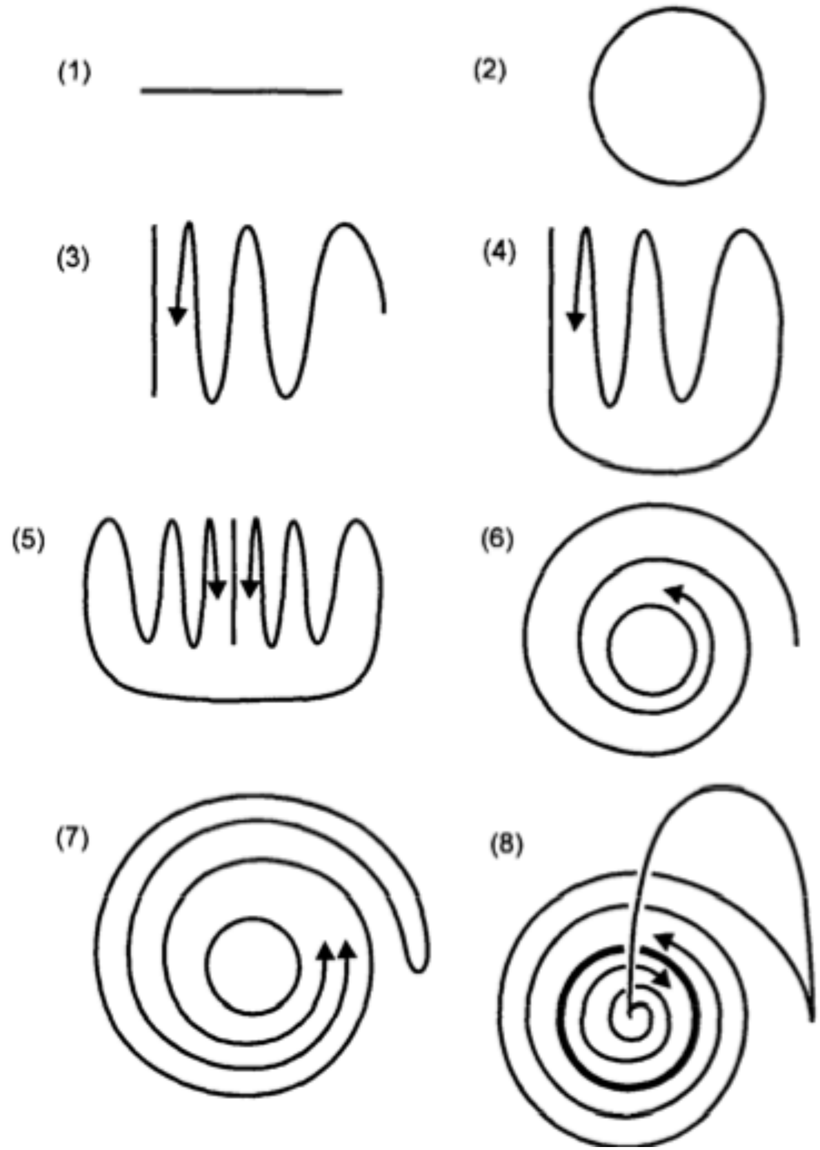
Hyperspaces and Cones

- Problem (I) has been widely study for the hyperspaces C(X) and not so much for Cn(X).

Hyperspaces and Cones

- **Theorem**(Rogers-Nadler 70´s) There are exactly 8 continua that are hereditarilly decomposable finite dimensional and satisfy that C(X)=Cone(X).

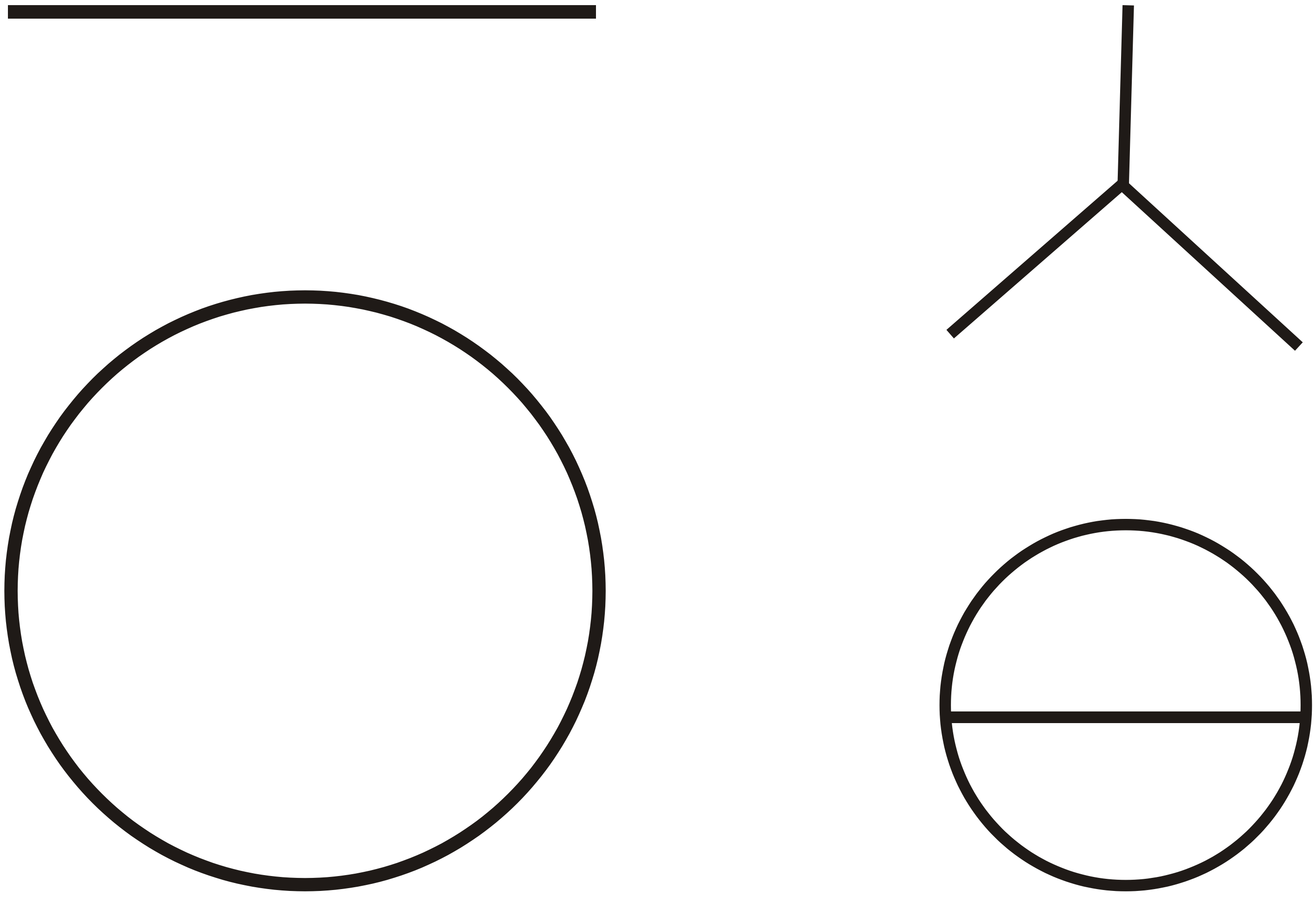
Cone(X)=C(X) (Rogers-Nadler)

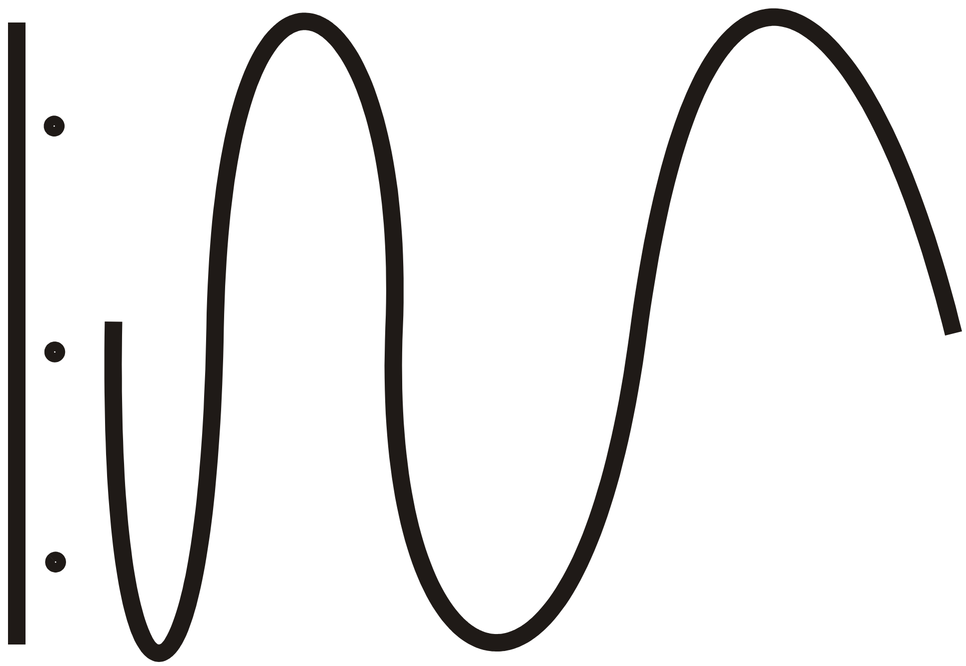


DEOMPOSABLE CONTINUA

- A continuum X is ***DECOMPOSABLE*** if there exist two proper nondegenerate subcontinua A, B of X, such that X = A U B

FINITE GRAPHS



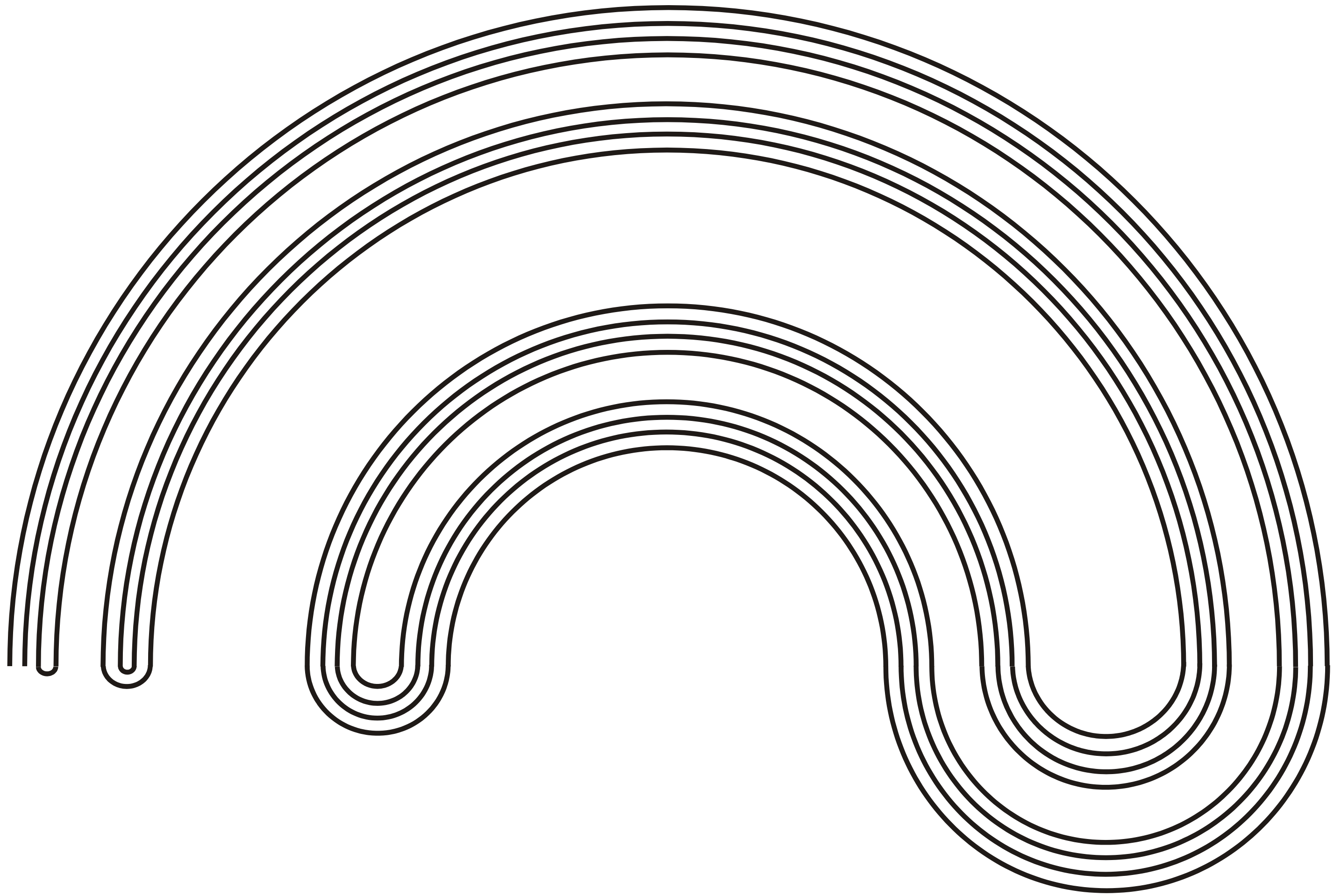


Sen($1/x$)-continuum

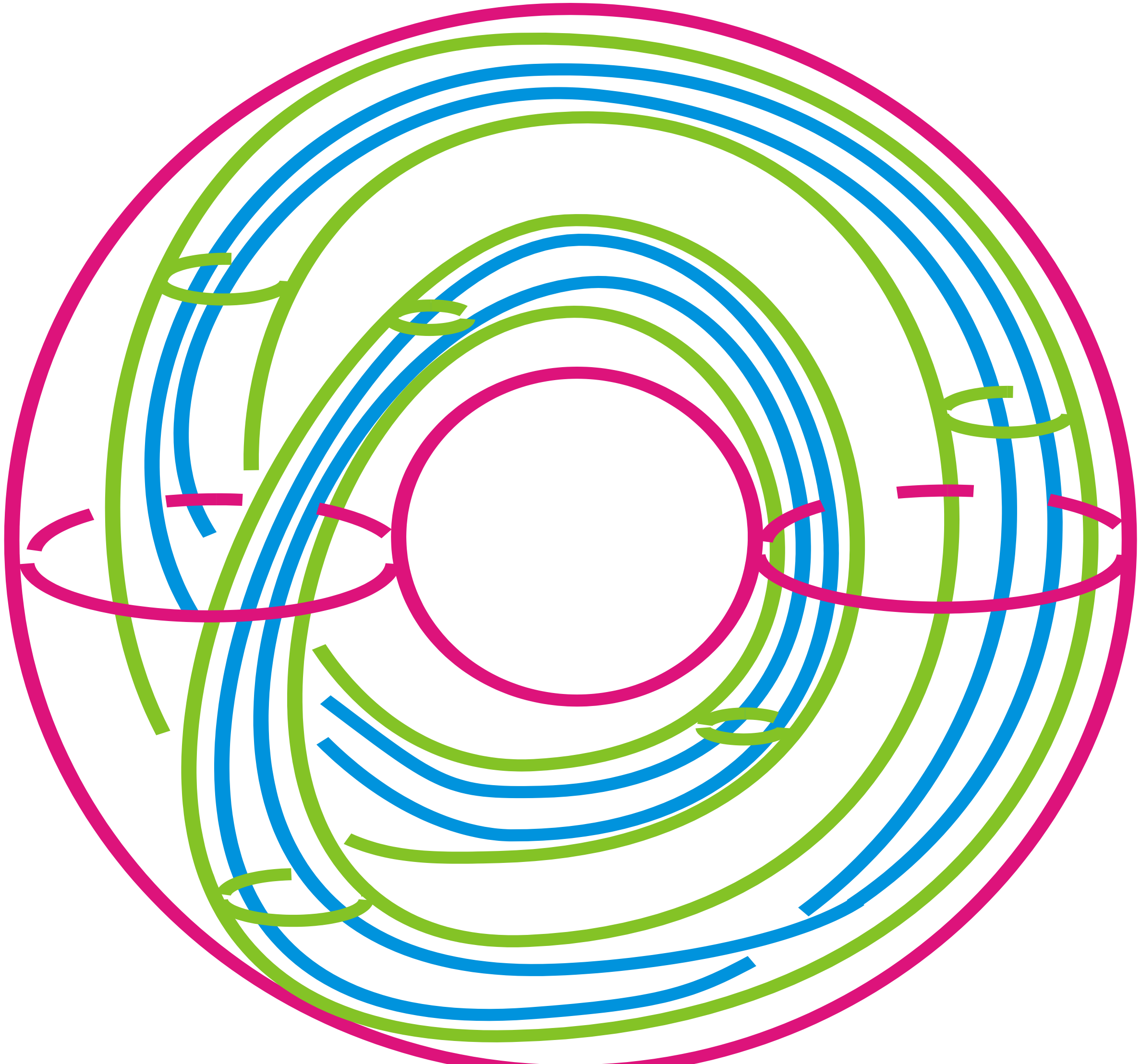
INDECOMPOSABLE CONTINUA

- A continuum X is ***INDECOMPOSABLE***, if it is not decomposable

Knaster Continuum



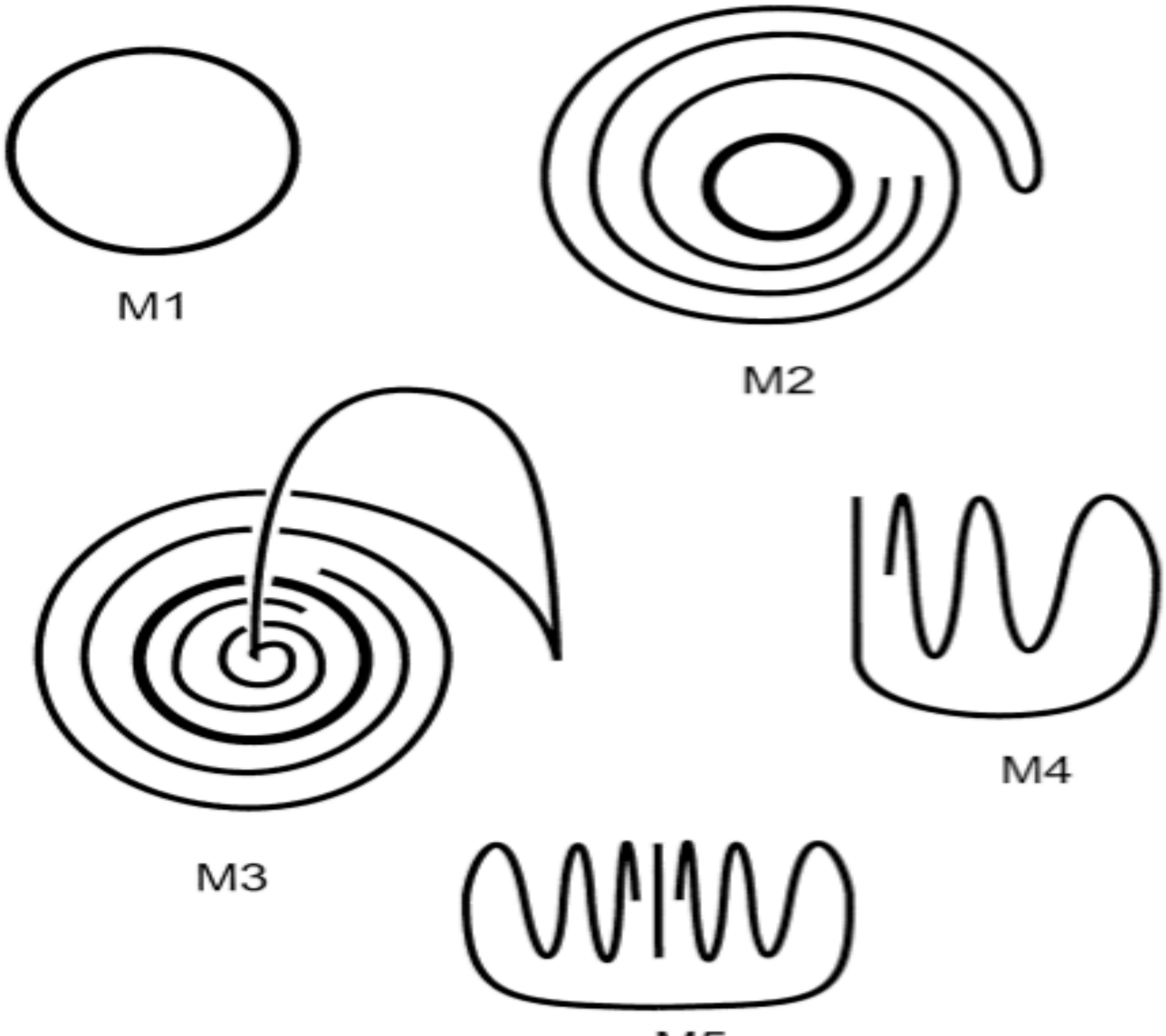
Solenoid



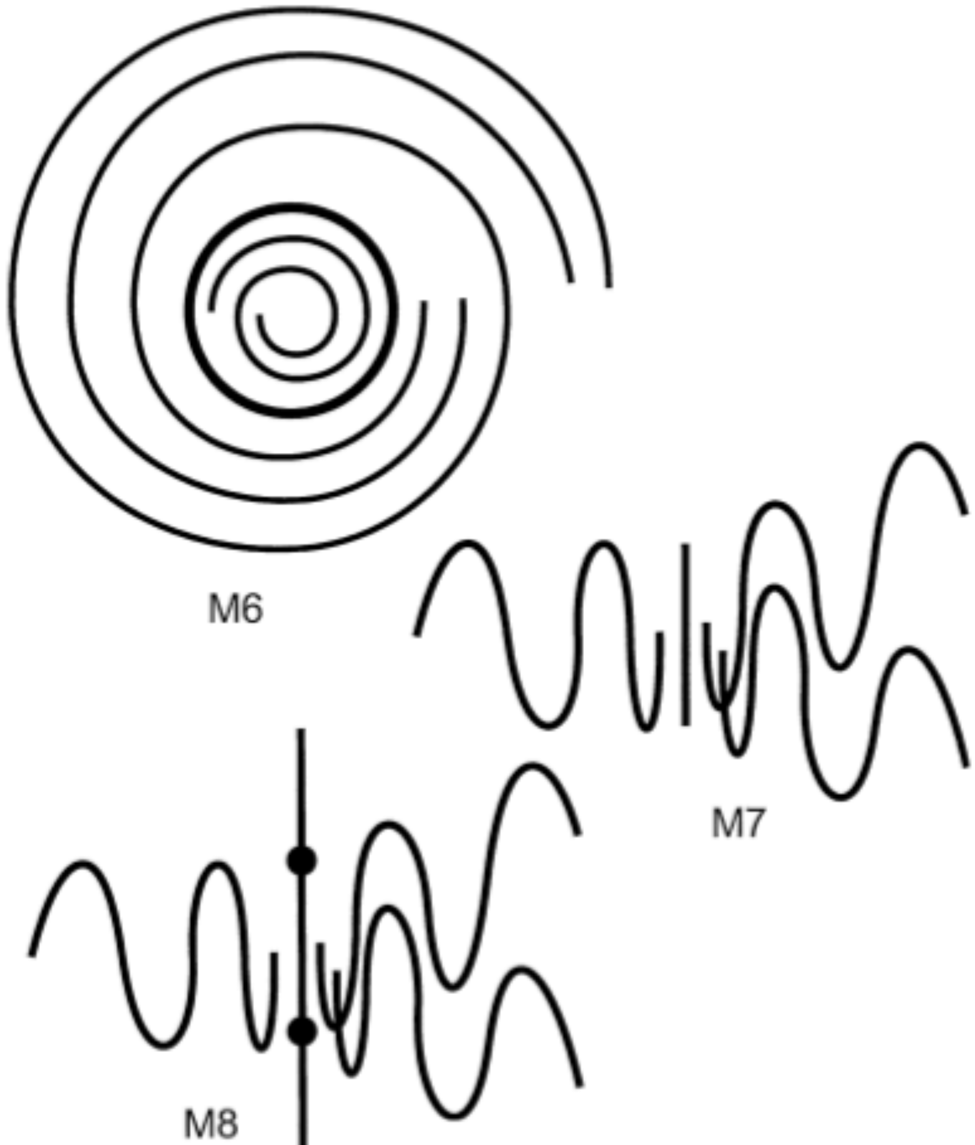
The Hyperspace C(X) and Cones

- **Theorem. (Illanes, López 2002)** *Let* X *be a finite dimensional* ***hereditarily decomposable*** *continuum. Then* C(X) *is a* cone *if and only if* X *is in one of the classes of continua described in* (M1) *to* (M10)

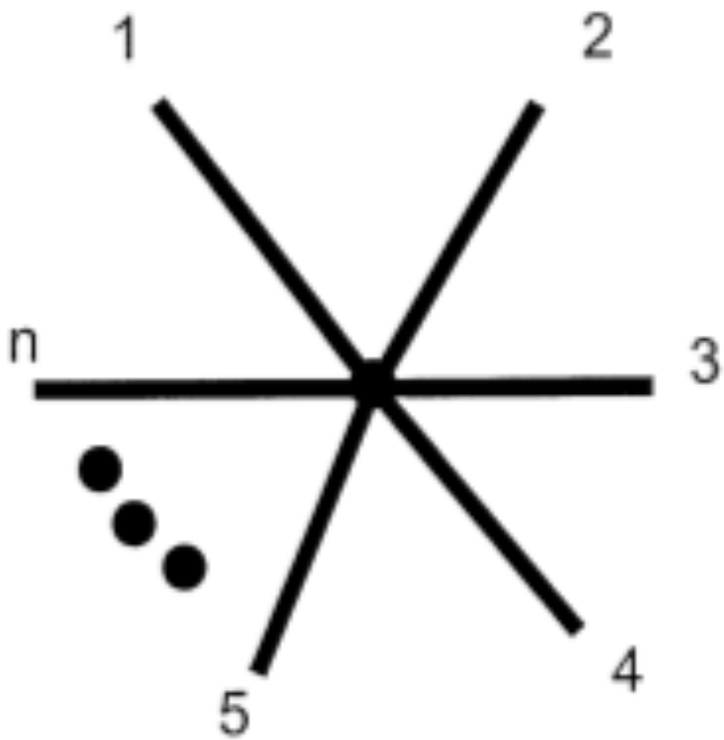
The Hyperspace C(X) and Cones



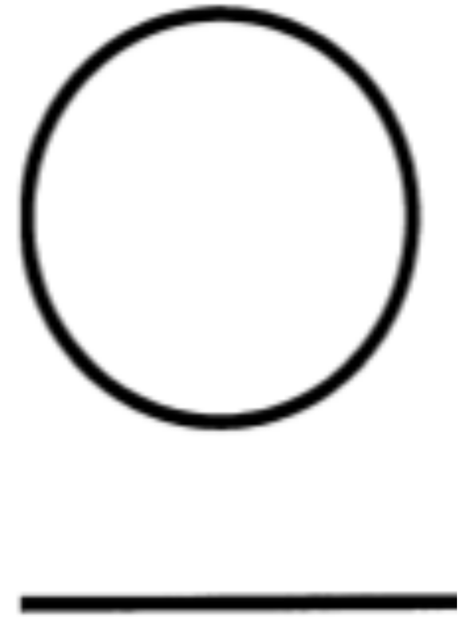
The Hyperspace C(X) and Cones



The Hyperspace C(X) and Cones



M9



M10

The Hyperspace C(X) and Cones

- **Theorem (Lopez 2002)** *Let* X *be a finite dimensional* ***non hereditarily decomposable*** *continuumm. Suppose that C(X) is a cone. Then there exists a* ***unique indecomposable subcontinuum Y*** *of X such that:*
- (a) C(Y) is a cone (cone=hyperspace property)
- (b) X − Y *is locally connected,*

- (c) $X - Y$ *has a finite number of components,*
- (d) *each component of* $X - Y$ *is homeomorphic either to* $[0,\infty)$ *or to the real line,*
- (e) Y is an arc continuum (all its proper subcontinua are arcs or points)

Cone=Hyperspace

- A continuum has the ***cone=hyperspace property*** provided that there exists a homeomorphism $h:C(X)\rightarrow Cone(X)$ such that $h(F1(X))=Base(Cone(X))$ and $h(X)=vertex(Cone(X))$.

Questions remaining for C(X)

- **Question 1**. Characterize all finite dimensional indecomposable continua with the cone=hyperspace property.

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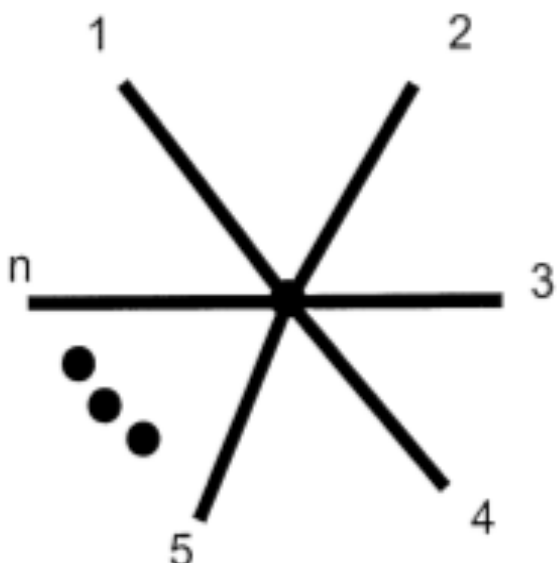
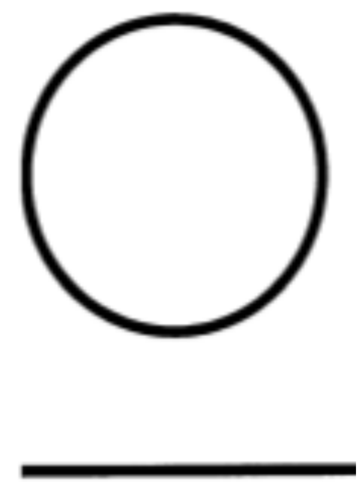
n-fold hyperspaces and cones

- **Theorem** (VMV 2004) *Let* X *be a* ***finite graph*** *If* Cn(X) *is a* cone *then* X *is an arc a circle or an n-od*





Questions remaining for Cn(X)

- **Question 2**. Is $C_3(S^1)$ a cone?
- **Question 3.** Is $C_n(S^1)$ a cone for $n \geq 3$?
- **Question 4**. Is $C_2(\mathrm{Sin}(1/x))$ a cone?
- **Question 5**. Is $C_2(\mathrm{Knaster})$ a cone?
- **Question 6**. Is $C_2(\mathrm{Solenoid})$ a cone?

Questions remaining for Cn(X)

- **Question 7.** Let X be a fan. Suppose that there exists $n \geq 2$ such that Cn(X) is a cone, does this imply that X is a cone?
- **Question 8.** Characterize finite dimensional continua X for which Cn(X) is a cone.

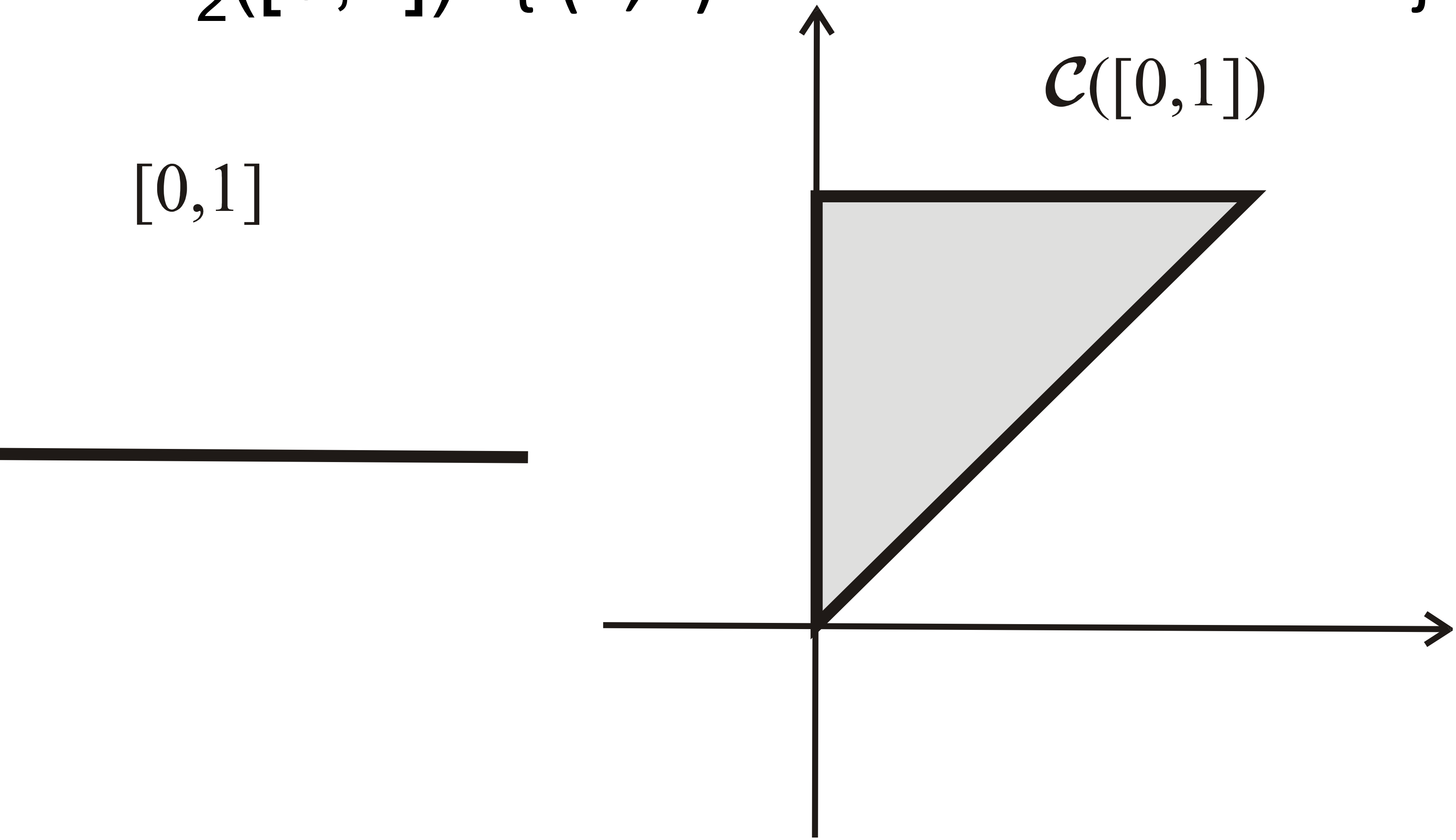
Symmetric Products and Cones

The structure of the hyperspaces C(X), Cn(X) is richer than the structure of Fn(X). An important difference is that C(X), Cn(X) are always arcwise connected and they are always locally connected at X.

Symmetric Products and Cones

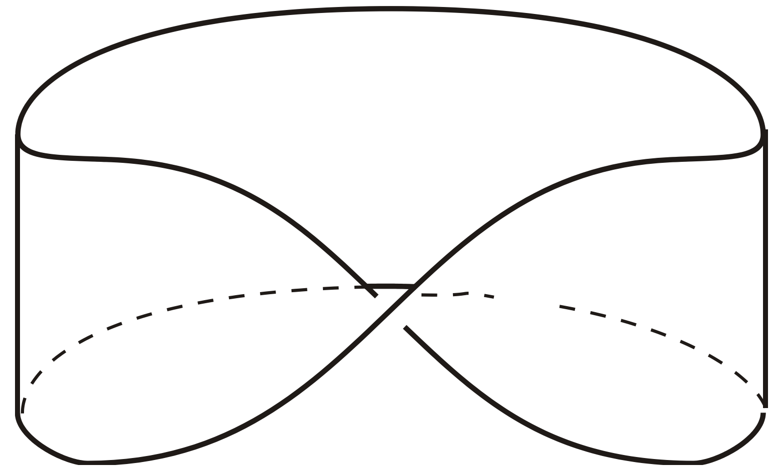
- On the other hand Fn(X) is arcwise connected if and only if X is arcwise connected and Fn(X) has not necessarily points of local connectedness.

$F_2([0,1])=\{ (a,b) \in R^2 : 0 \le a \le b \le 1 \}$

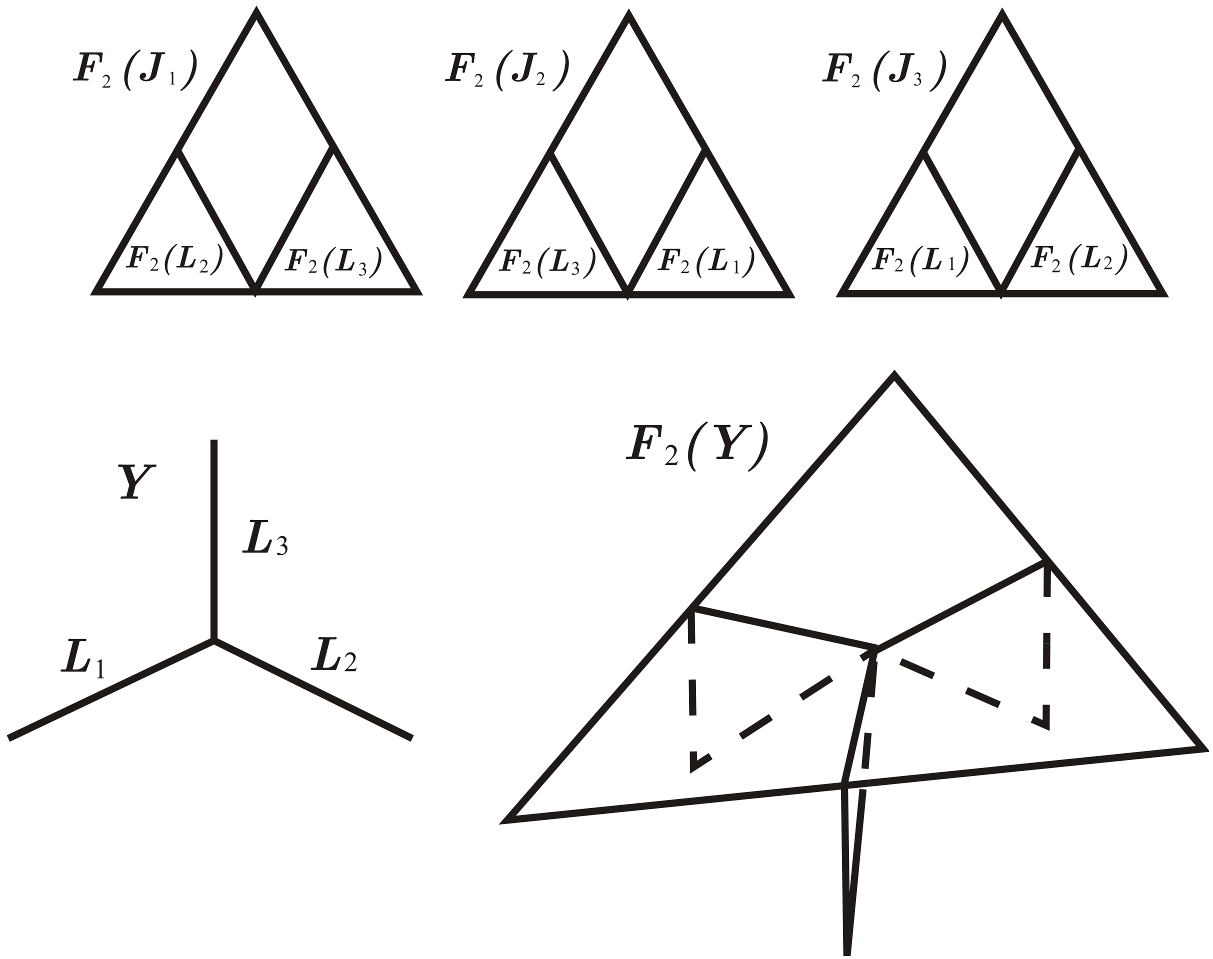


$F_2(S^1)$

- $F_2(S^1)$ is a Möbius strip



F2(T)



Symmetric Products and Cones

- **Theorem(**_VMV-Illanes 2015_**)** Suppose that the continuum X is a cone. Then each of the hyperspaces 2^X, C(X), $C_n(X)$ and $F_n(X)$ is a cone.

Symmetric Products and Cones

- **Theorem** (*VMV-Illanes 2015*) Let X be a ***finite graph*** . Then the following are equivalent

(a) X is a cone

(b) Fn(X) is a cone for every n≥2 and

(c) Fn(X) is a cone for some n≥2

Symmetric Products and Cones

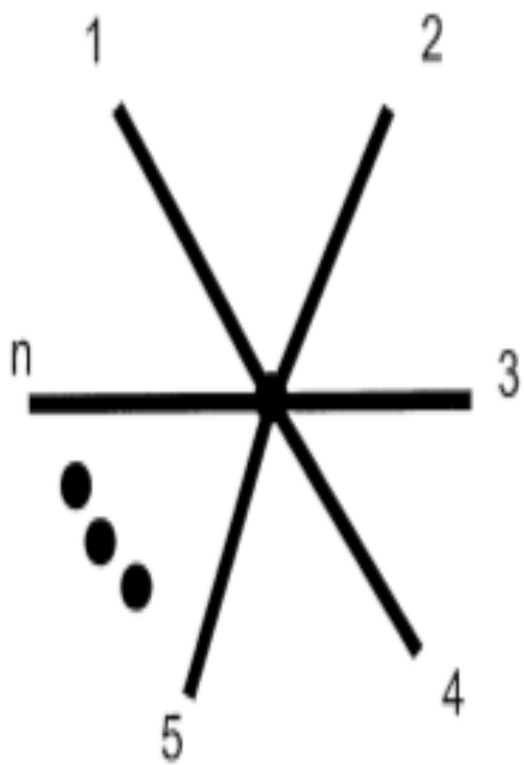
- **Theorem** (*VMV-Illanes 2015*) Let X be a ***fan***. Then the following are equivalent

(a) X is a cone

(b) Fn(X) is a cone for every n≥2 and

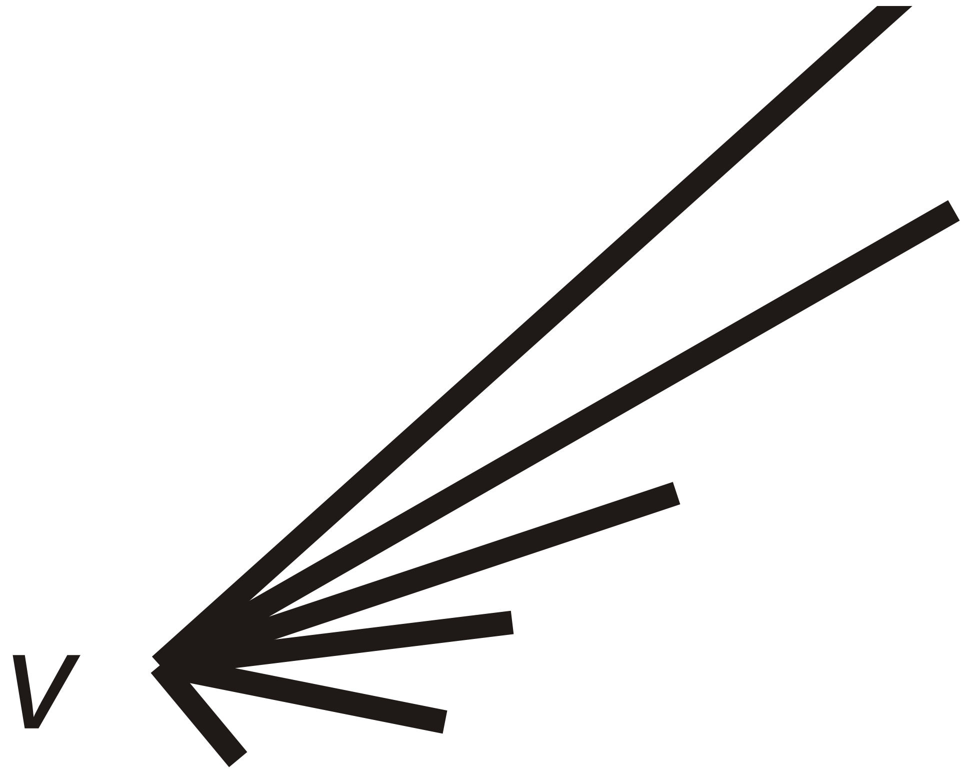
(c) Fn(X) is a cone for some n≥2

Finite graphs that are fans and Cones



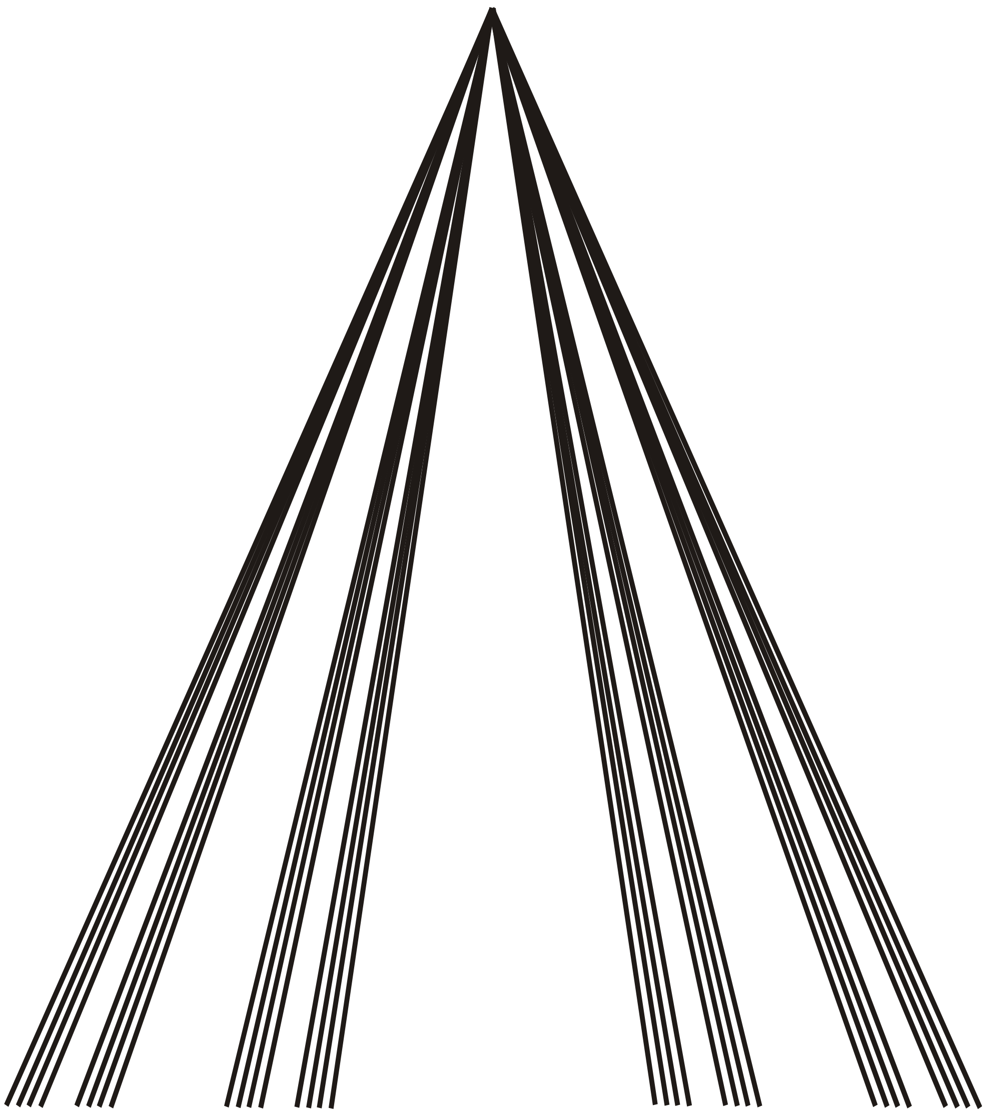
$[0,1]$

Fω is a fan, but not a cone



Harmonic Fan

Cantor Fan



Symmetric Products and Cones

- **Question 9**. Suppose that X is a continuum such that for some $n \geq 2$, Fn(X) is a cone, must X itself be a cone?
- **Question 10**. Suppose that X is a dendroid such that for some $n \geq 2$, Fn(X) is a cone, must X itself be a cone?

PROBLEM II
UNIQUENESS OF HYPERSPACES

Uniqueness of Hyperspaces

- For a metric continuum X we say that X ***has unique hyperspace K(X)*** provided that, if Y is a continuum and K(X) is homeomorphic to K(Y), then X is homeomorphic to Y .

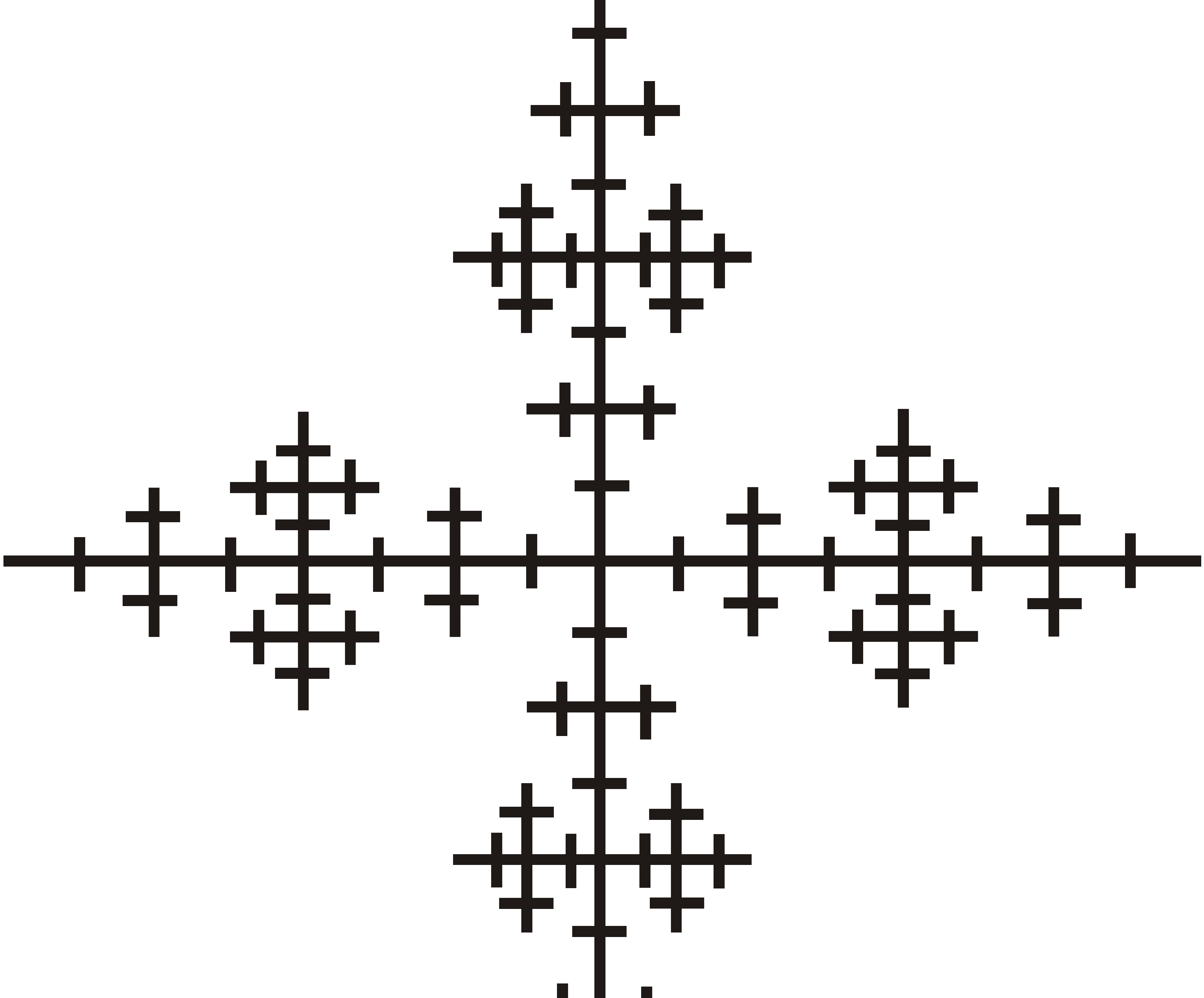
Uniqueness of Hyperspaces

- **Theorem 1** (Curtis-Schori 1978). If X is a ***locally connected continuum***, then 2^X is homeomorphic to the Hilbert cube.

Uniqueness of Hyperspaces

Theorem 2. (Curtis-Schori 1978) For a continuum X, the following are equivalent.

- (a) X ***is locally connected and each arc in X has empty interior***,
- (b) C(X) is homeomorphic to the Hilbert cube
- (c) Cn(X) is homeomorphic to the Hilbert cube for each n.



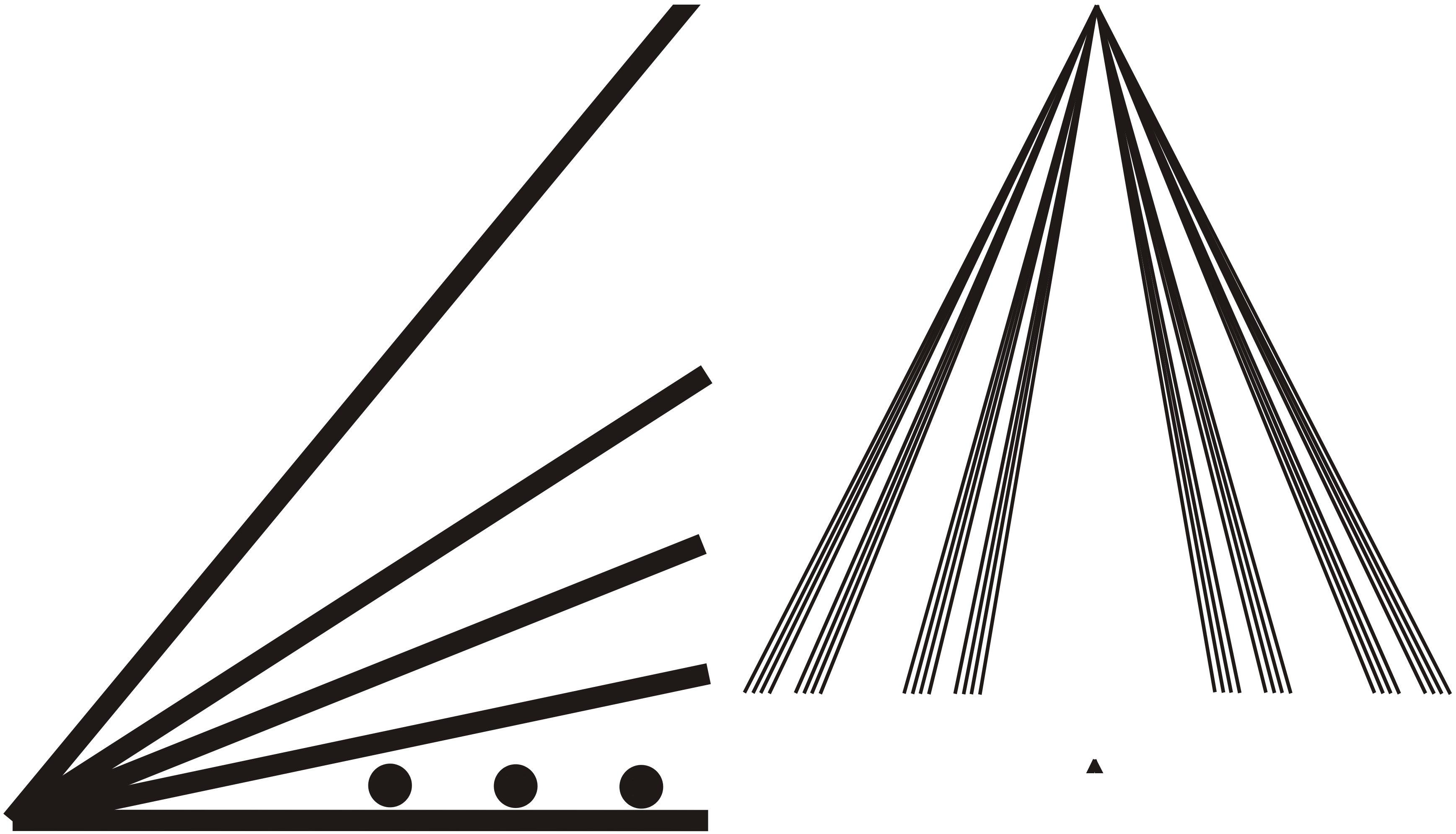
Uniqueness of Hyperspaces

- **Theorem 3.** (Duda 1968-1970). Finite graphs G, different from an arc and a simple closed curve have unique hyperspace C(G).

Uniqueness of Hyperspaces

- **Theorem 4.**(Illanes 2003) Finite graphs G have unique hyperspace Cn(G) for each $n \geq 2$.
- **Theorem 5** (Eberhart-Nadler 1979). If X is a smooth fan with infinitely many end points, then X does not have unique hyperspace C(X).

Uniqueness of Hyperspaces

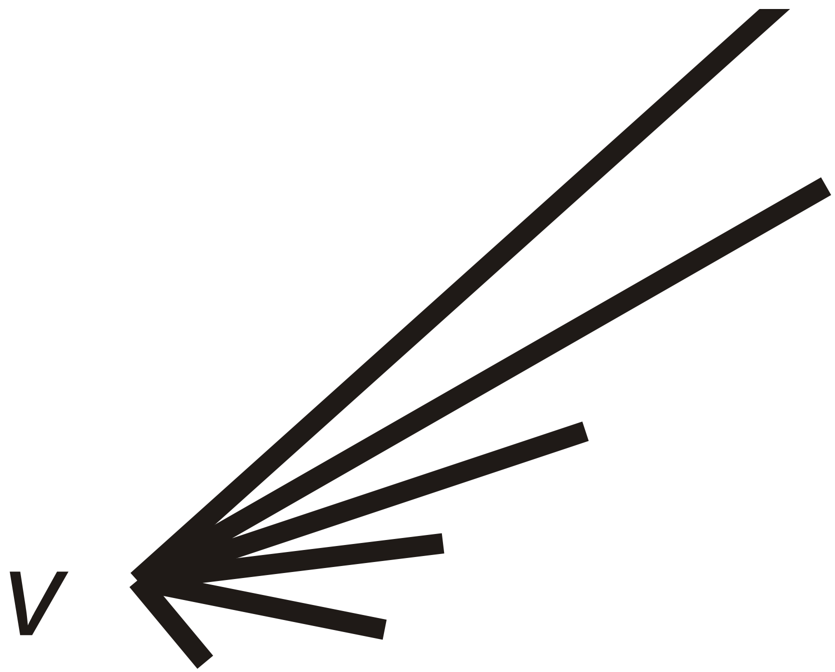


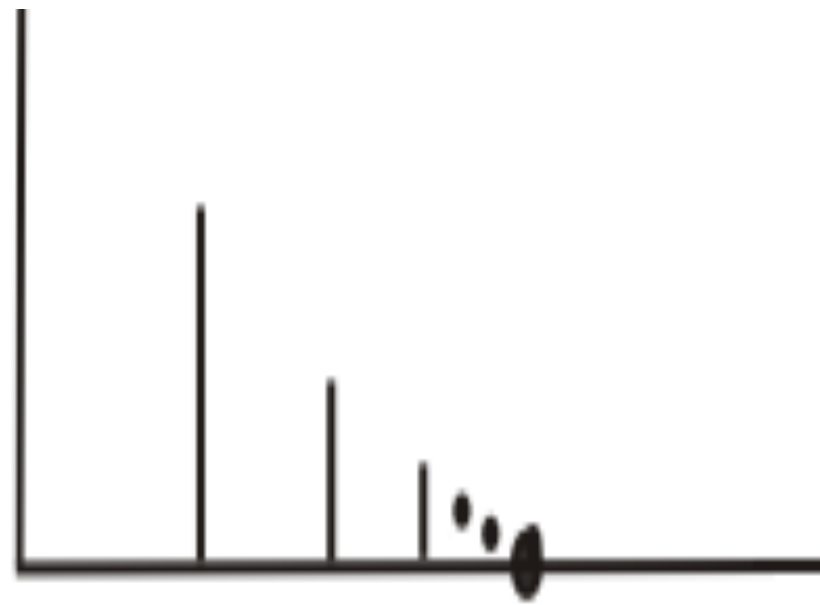
Uniqueness of Hyperspaces

- A ***dendrite*** is a locally connected continuum without simple closed curves. Define
- $\mathcal{D}$ = {X : X is a dendrite with closed set of end points}.

UNIQUENESS OF HYPERSPACES

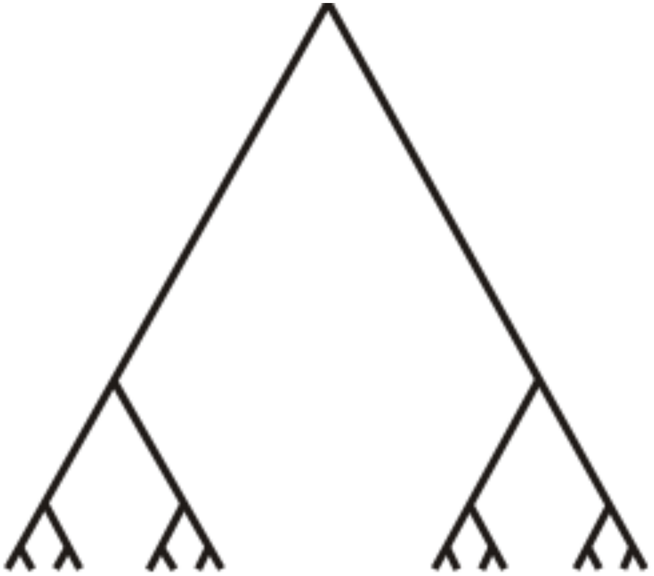
- It is known that a dendrite $X \in \mathcal{D}$ if and only if X does not contain neither a copy of $F\omega$ nor a copy of the enlarged null comb.





UNIQUENESS OF HYPERSPACES

- Gehman dendrite is a dendrite in class $\mathcal{D}$. For this dendrite, the set of end points is the Cantor set.



Uniqueness of Hyperspaces

- **Theorem 6** (Herrera-Illanes-Macìas-Romero and López). Let X $\in \mathcal{D}$. Then
- (a) X has unique hyperspace Cn(X) for each n≥2,
  (b) if X is not an arc, then X has unique hyperspace C(X).

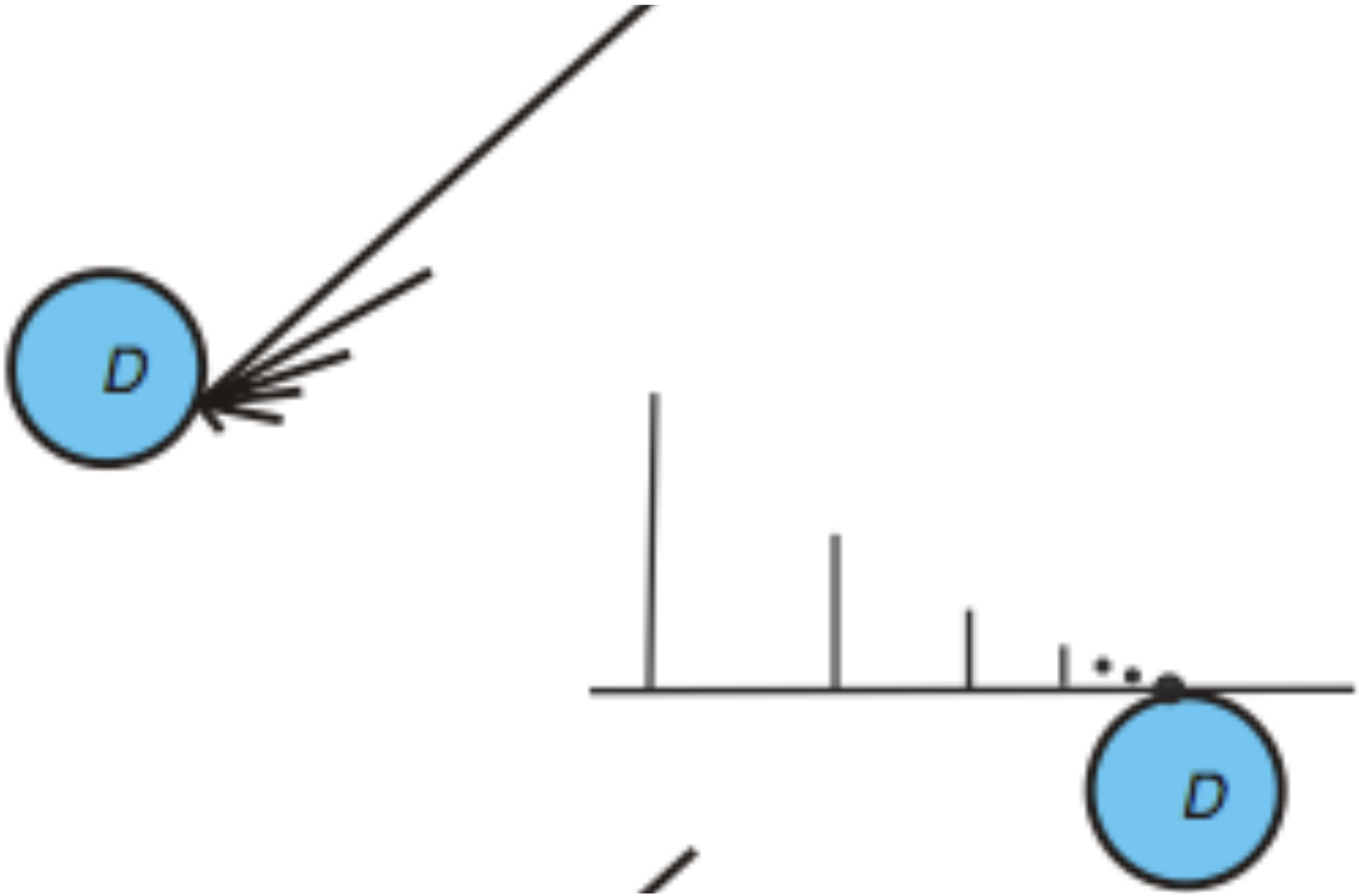
UNIQUENESS OF HYPERSPACES

- **Theorem 7.** (Herrera-Macias 2011). Continua with a base of neighborhoods belonging to class $\mathcal{D}$ have unique hyperspace Cn(X) for all n$\neq$ 2.

UNIQUENESS OF HYPERSPACES

- **Theorem 8** (Acosta, Herrera 2010). If a dendrite X does not belong to $\mathcal{D}$, then X does not have unique hyperspace C(X).

UNIQUENESS OF HYPERSPACES



UNIQUENESS OF HYPERSPACES

- **Example** (Hernandez-G, Illanes,VMV 2013)There exists a dendrite containing the extended null comb and having unique hyperspace $C_2(X)$

- A locally connected continuum X is ***almost framed*** provided that

$$\bigcup \{J \subset X : J \text{ is a free arc in } X\}$$

is dense in X.

- G(X) = {p ∈ X : p has a neighborhood K in X such that K is a finite graph}.
- A locally connected continuum X is almost framed if and only if G(X) is dense in X.

UNIQUENESS OF HYPERSPACES

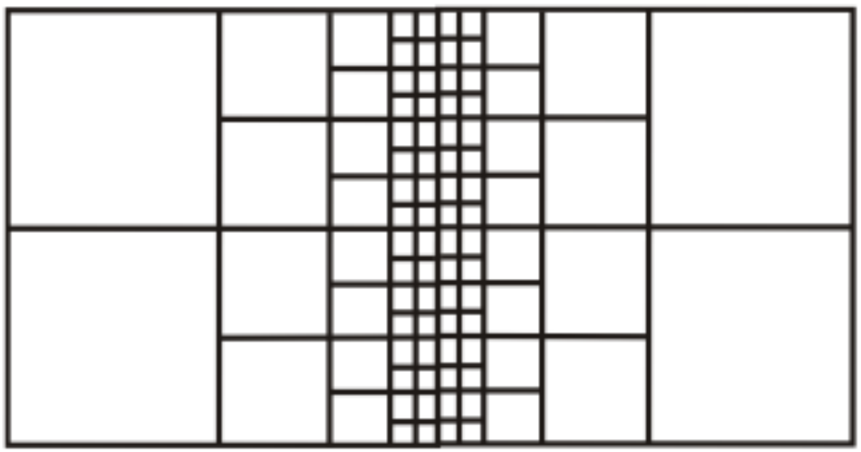
- A continuum X is **framed** if:

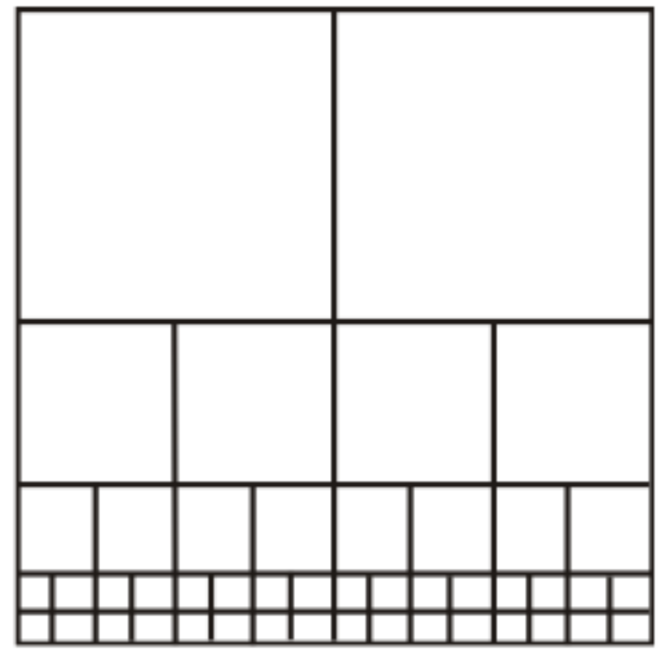
(i) it is not a simple closed curve,

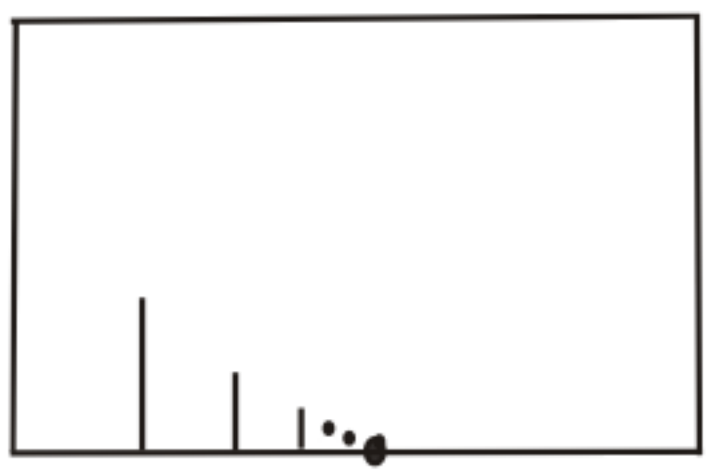
(ii) is almost framed and

(iii) has a base of neighborhoods $\mathcal{B}$ such that for each $U \in \mathcal{B}$, $U \cap G(X)$ is connected.

- Finite graphs, dendrites in class $\mathcal{D}$ and locally class-$\mathcal{D}$ dendrites are framed continua.







Hernandez-G, Illanes, VMV 2013

- **Theorem 9** Framed continua have unique hyperspace $C_n(X)$ for all $n \in N$.
- **Theorem 10** If X is a locally connected continuum and X is not almost framed, then X does not have unique hyperspace $C_n(X)$ for each $n \in N$.
- **Theorem 11** If X is almost framed and X-G(X) is not connected, then X does not have unique hyperspace C(X).

UNIQUENESS OF HYPERSPACES

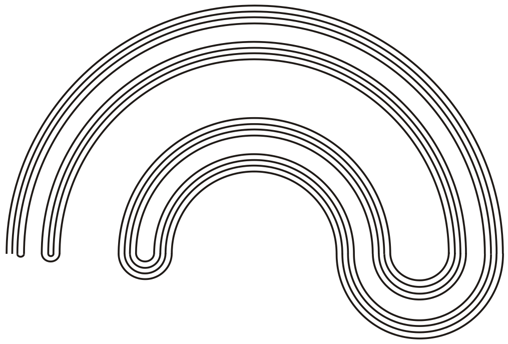
- **Theorem 12** (Acosta 2002)). If X is a ***compactification of the ray*** and X is not an arc, then X has unique hyperspace C(X).
- If X is a ***compactification of the real line***, X is not an arc and its ***remainder is disconnected***, then X has unique hyperspace C(X).

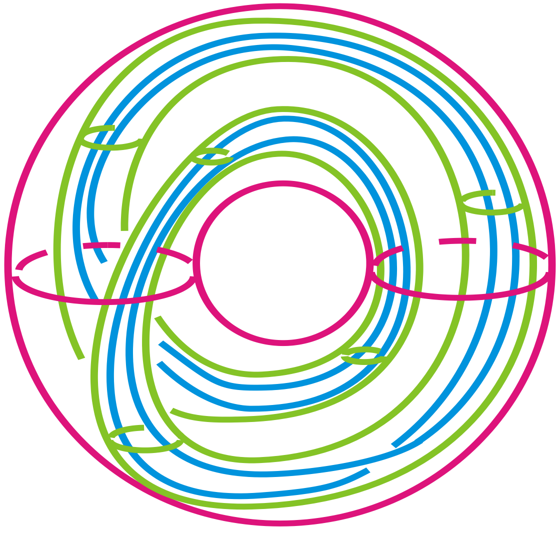
UNIQUENESS OF HYPERSPACES

- **Theorem 13** (Macas 2002) If X is a hereditarily indecomposable continuum, then X has unique hyperspaces 2^X and $C_n(X)$ for all $n \in N$.
- **Theorem 14** (Acosta 2002) Indecomposable arc continua have unique hyperspace C(X).

UNIQUENESS OF HYPERSPACES

- **Theorem 15** (HG-I-MV 2013) Indecomposable arc continua X have unique hyperspace Cn(X), for each n ≠ 2, the case n = 2 remains unsolved.





Questions on n-fold hyperspaces

- **Question 1**. If X is a smooth fan with infinitely many end points, does X not have unique hyperspace Cn(X) for $n \geq 2$?
- **Question 2**. Characterize locally connected continua X which have unique hyperspace Cn(X).
- **Question 3**. Do compactifications of the ray have unique hyperspace Cn(X) for each $n \geq 2$?

Questions on n-fold hyperspaces

- **Question 4.** Do compactifications of the real line with disconnected remainder have unique hyperspace $C_2(X)$?
- **Question 5**. Let X be a compactification of the real line. Does X have unique hyperspace $C_2(X)$?

- **Question 6.** Find more classes of continua X having unique hyperspace 2^X
- **Question 7**. Have indecomposable arc continua X unique hyperspaces 2^X and $C_2(X)$?
- **Question 8**. Do there exist two non-homeomorphic fans X and Y such that $C_2(X)$ and $C_2(Y)$ are homeomorphic?

- **Question 9**. Let $n \geq 2$ and X and Y be smooth fans such that Cn(X) is homeomorphic to Cn(Y). Does it follow that X is homeomorphic to Y ?
- **Question 10**. Let X and Y be smooth fans such that 2^X is homeomorphic to 2^Y and X has infinitely many end points. Does it follow that X is homeomorphic to Y ?

UNIQUENESS OF SYMMETRIC PRODUCTS

- **Theorem 16** (HG-I-MV).
- (a) (Acosta-Herrera-Lopez) Finite graphs G have unique hyperspace Fn(G) for every $n \in \mathbb{N}$.
- (b) (HG-I-MV) Dendrites $X \in \mathcal{D}$ have unique hyperspace Fn(X).

UNIQUENESS OF SYMMETRIC PRODUCTS

- **Theorem 17** (Illanes-J. Martinez 2009).
- (a) Compactifications X of the ray [0,1) have unique hyperspace $F_n(X)$ for each n ≠ 3.
- (b) Compactifications X of the ray [0,1) such that the remainder is an ANR have unique hyperspace $F_3(X)$.

UNIQUENESS OF SYMMETRIC PRODUCTS

- **Theorem 18** (Illanes-Castañeda-Anaya 2013). The following type of continua:
- Indecomposable arc continua,
- Fans or
- Arcwise connected continua with exactly only one ramification p

have unique hyperspace $F_2(X)$.

Rigidity of Hyperspaces

- A useful technique is to find a topological property that characterizes the elements of $F_1(X)$ in the hyperspace $K(X)$.
- When this is possible the hyperspace $K(X)$ is rigid, so both topics are closely related.

Rigidity of Hyperspaces

- A hyperspace $K(X)$ of X is said to be ***rigid*** provided that for every homeomorphism

$$h : K(X) \to K(X)$$

we have that

$$h(F_1(X)) = F_1(X).$$

- A ***wire*** in a continuum X is a subset α of X such that α is a component of an open subset of X and is homeomorphic to one of the spaces (0,1), [0,1), [0,1] or S^1
- Given a continuum X, let

  $W(X) = \{\alpha \subset X : \alpha \text{ is a wire in } X\}$.
- The ***continuum X is*** said to be ***wired*** provided that ***W(X) is dense in X***.

Rigidity and uniqueness of Symmetric Products

- HG-MV 2013
- **Theorem 19.** *Let $n \geq 4$ and let X be a wired continuum. Then:*

  (a) X has unique hyperspace $Fn(X)$

  (b) *$Fn(X)$ is rigid.*

- **Corollary 20** Compactifications of the ray, Smooth Fans, indecomposable arc continua are wired continua.

UNIQUENESS AND RIGIDITY OF SYMMETRIC PRODUCTS (HG-MV 2013)

- **Theorem 21.** *If a continuum X contains a tail, then* $F_2(X)$ *is not rigid.*
- **Theorem 22.** *Let X be an almost meshed. Then* $F_2(X)$ *is rigid if and only if X does not contain tails.*
- **Corollary 23.** *A finite graph X has rigid hyperspace* $F_2(X)$ *if and only if X does not have end points.*
- **Theorem 24.** *If a continuum X contains a free arc, then* $F_3(X)$ *is not rigid.*

Questions on Symmetric Products

- **Question 11**. Have all dendrites X unique hyperspace $F_n(X)$?
- **Question 12**. Have all compactifications X of the ray [0,1) unique hyperspace $F_3(X)$?
- **Question 13**. Have all chainable (circle-like) continua X unique hyperspace $F_n(X)$?

Questions on Symmetric Products

- **Question 14**. Have all fans X unique hyperspace $F_n(X)$?

- **Question 15**. Have all indecomposable arc continua X unique hyperspace $F_3(X)$?

Questions on Symmetric Products

- **Question 16**. Does there exist a finite dimensional continuum X without unique hyperspace $F_n(X)$?
- **Question 17**. Do hereditarily indecomposable continua X have unique hyperspace $F_2(X)$?
- **Question 18**. Does the Pseudo-arc have unique hyperspace $F_2(X)$?

PROBLEM III
HOMOGENEITY DEGREE OF HYPERSPACES

Homogeneity Degree

The ***homogeneity degree***, ***hd(X)***, of X is the number of orbits in X for the action of the group of homeomorphisms of X onto itself.

Given a continuum X, let ***H(X)*** denote the group of homeomorphisms of X onto itself.

Homogeneity Degree

An ***orbit*** in X is a class of the equivalence relation in X given by p is equivalent to q if there exists h in H(X) such that h(p)=q.

- The ***homogeneity degree, hd(X),*** of the continuum X is defined as

hd(X)=number of orbits in X

Homogeneity Degree

- When ***hd(X)=n*** the continuum X is known to ***be 1/n-homogeneous***
- and when ***hd(X)=1***, X is ***homogeneous.***

Previous Results

- In 2008 Pellicer studied continua for which hd(F2(X))=2.

- **Theorem** (2015 I. Calderón, R. Hernández-Gutiérrez and A. Illanes)

  If P is the pseudo-arc, then $hd(F_2(P))=3$

- **Theorem**(HG-MV 2015) Let X be an m-manifold without boundary and n a natural number. Then

(a) If either $m=2$ and $n\neq 2$ or $m=1$ and $n\neq 3$ then $hd(Fn(X))=n$.

(b) If $m=2$ (X is a surface), then $hd(F_2(X))=1$ and

(c) If $m=1$ (X is a simple closed curve) and $n=3$, then $hd(F_n(X))=1$.

- **Theorem (HG-MV 2015)**. Let n be a natural number. Then:

(a) If $n \geq 4$, then hd(Fn([0,1])=2n, and

(b) If $n \in \{2,3\}$, then hd(Fn([0,1])=2.

THANKYOU