Cotorsion-free groups from a topological viewpoint

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joint work with
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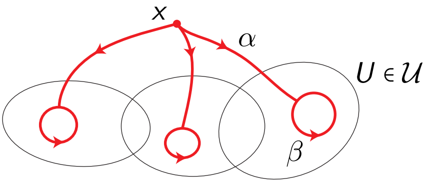
Definition

For an open cover $\mathcal{U}$ of a path-connected space X and $x \in X$, $\pi(\mathcal{U}, x) \leqslant \pi_1(X, x)$ is generated by all elements $[\alpha\beta\alpha^-]$ with $\alpha : ([0,1], 0) \to (X, x)$, $\beta : ([0,1], \{0,1\}) \to (U, \alpha(1))$, $U \in \mathcal{U}$.



Generalized covering spaces

- Asphericity criteria
- "Cayley graph" for $\pi_1(\mathbb{M})$ of the Menger curve $\mathbb{M}$

⋮

Generalized slender groups

- Theory of free σ-products
- Classification of homotopy types of 1-dim spaces by π_1

⋮

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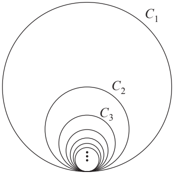

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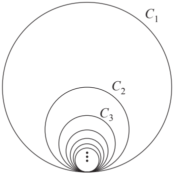

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Definition: $\pi^s(X,x) = \bigcap_{\mathcal{U}} \pi(\mathcal{U},x)$ **(Spanier group)**

Theorem [F-Zastrow 2007]

There exists a **generalized covering** $p:\widetilde{X}\to X$ w.r.t. $\pi^s(X,x)$:

(1) $\widetilde{X}$ is path connected (pc) and locally path connected (lpc).

(2) $p_\#:\pi_1(\widetilde{X},\widetilde{x})\to\pi_1(X,x)$ is a monomorphism onto $\pi^s(X,x)$.

(3)

$$\begin{array}{ccc} & & (\widetilde{X},\widetilde{x}) \\ & \overset{\exists!\,\widetilde{f}}{\nearrow} & \big\downarrow p \\ (Y^{pc,lpc},y) & \xrightarrow{\ \forall f\ } & (X,x) \end{array} \quad \Longleftrightarrow \quad f_\#\pi_1(Y,y)\leqslant p_\#\pi_1(\widetilde{X},\widetilde{x})$$

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Examples with $\pi^s(X, x) = 1$ include:

- 1-dimensional spaces [Eda-Kawamura 1998]
- subsets of surfaces [F-Zastrow 2005]
- certain "trees of manifolds" [F-Guilbault 2005]

Definition

An abelian group A is called **slender** if for every homomorphism $h : \mathbb{Z}^{\mathbb{N}} \to A$, $\exists\ n \in \mathbb{N}\ \forall c_n, c_{n+1}, \dots \in \mathbb{Z}$: $h(0, \dots, 0, c_n, c_{n+1}, \dots) = 0$.

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A group G is called **n-slender** if for every homomorphism $h : \pi_1(\mathbb{H}, *) \to G$, $\exists\ n \in \mathbb{N}\ \forall \gamma \subseteq \bigcup_{k=n}^{\infty} C_k$: $h([\gamma]) = 1$.

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Examples: Free groups are n-slender [Higman, Griffiths 1952-56]. Certain HNN extensions of n-slender groups are n-slender, including the Baumslag-Solitar groups [Nakamura 2015].

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Theorems [Eda 1992, 2005]

(1) An abelian group A is n-slender $\Leftrightarrow$ A is slender.

(2) A group G is n-slender $\Leftrightarrow$ for every Peano continuum X and every homomorphism $h : \pi_1(X, x) \to G$, $\exists\, \mathcal{U} : h(\pi(\mathcal{U}, x)) = 1$.

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A group G is **homomorphically Hausdorff** relative to a space X if for every homomorphism $h : \pi_1(X, x) \to G$, $\bigcap_{\mathcal{U}} h(\pi(\mathcal{U}, x)) = 1$.

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Terminology: $\pi_1(X,x)$ is a topological group with basis

$$\{g\pi(\mathcal{U},x) \mid g \in \pi_1(X,x),\ \mathcal{U} \in Cov(X)\}.$$

Consider $K = h(\pi_1(X,x)) \leqslant G$ as the quotient of $h : \pi_1(X,x) \to K$.

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Examples of residually n-slender groups include π_1 of 1-dimensional spaces, planar sets, the Pontryagin surface $\prod_2$, and the Pontryagin sphere $\varprojlim \left(T^2 \leftarrow T^2 \# T^2 \leftarrow T^2 \# T^2 \# T^2 \leftarrow \cdots \right)$.

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(4) $\pi^s(X,x) = 1 \Rightarrow X$ is homotopically Hausdorff.

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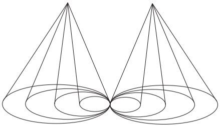

$\mathbb{G}$ is **not** homotopically Hausdorff.

Main Theorem

For an abelian group A, the following are equivalent:

(a) A is cotorsion-free.

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A is slender $\Leftrightarrow$ is cotorsion-free and $\mathbb{Z}^{\mathbb{N}} \nleqslant A$.

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Given $\sum_{i=1}^{\infty} i! u_i = (u_1 + 2!\mathbb{Z}, u_1 + 2!u_2 + 3!\mathbb{Z}, u_1 + 2!u_2 + 3!u_3 + 4!\mathbb{Z}, \dots)$

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find $[\ell] \in \pi_1(X, x)$ with $\quad n! \mid \left(h([\ell]) - \sum_{i=1}^{n-1} i!u_i a \right)$ in $A \quad \forall n \geqslant 2$.

Define $\phi(\sum_{i=1}^{\infty} i!u_i) = h([\ell])$. (Well-defined: $\bigcap_{n \in \mathbb{N}} n!A = \{0\}$.)

If $u_1 = 1$ and $u_i = 0$ for $i \geqslant 2$, then $\phi(\sum_{i=1}^{\infty} i!u_i) = a$.

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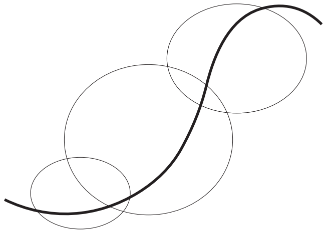

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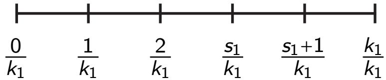

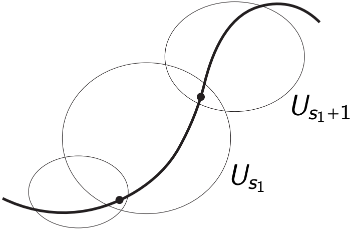

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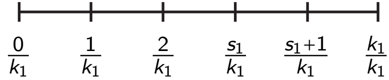

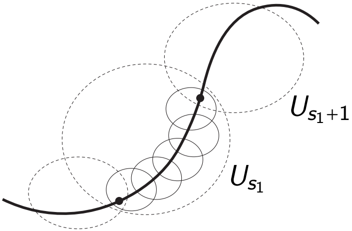

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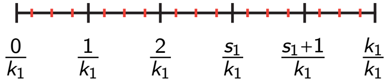

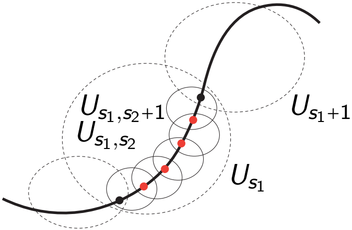

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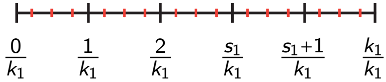

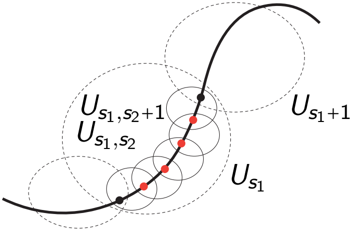

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and subdivision points $a_{s_1,s_2,\ldots,s_n} = \sum_{i=1}^{n} \frac{s_i}{\prod_{j=1}^{i} k_j} \in [0,1]$ such that

(1) $U_{s_1,s_2,\ldots,s_{n-1},s_n} \subseteq U_{s_1,s_2,\ldots,s_{n-1}}$

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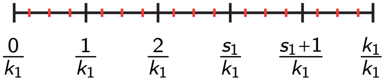

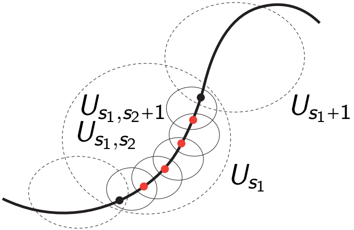

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Put $S_n = \{(s_1,s_2,\dots,s_n) \mid 0 \leqslant s_i < k_i\}$.

Since $u_n a \in h(\pi(\mathcal{U}_n, x))$ we have $u_n a = h(\prod_{s \in S_n} \prod_{i=1}^{r_s} [\alpha_{s,i} \beta_{s,i} \alpha_{s,i}^{-}]) \in A$

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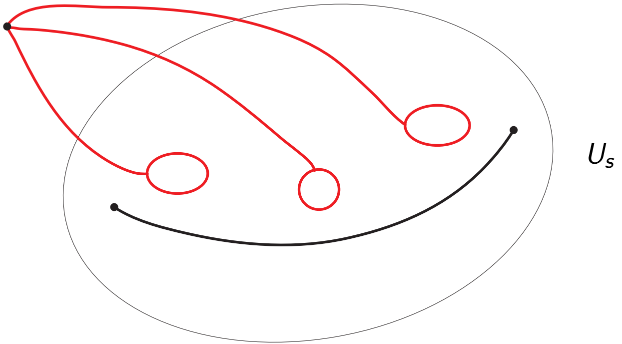

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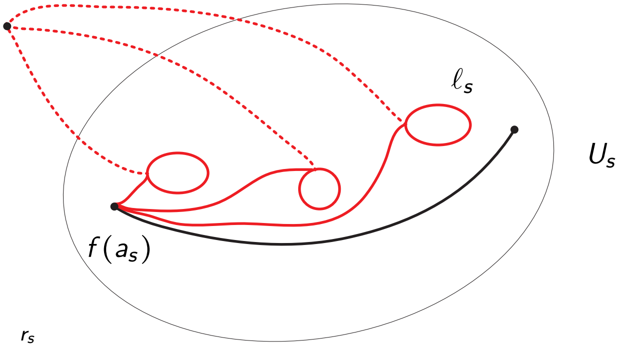
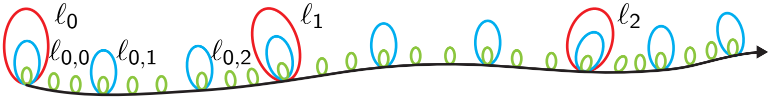


Put $\ell_s = \prod_{i=1}^{r_s} \gamma_{s,i}\beta_{s,i}\gamma^-_{s,i}$ with $\gamma_{s,i} : ([0,1],0) \to (U_s, f(a_s))$.

Then $u_n a = \widetilde{h}(\sum_{s\in S_n} [\ell_s])$ where $h : \pi_1(X,x) \to H_1(X) \xrightarrow{\widetilde{h}} A$.


ℓ_0
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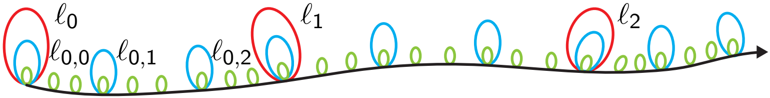
ℓ_1
ℓ_2



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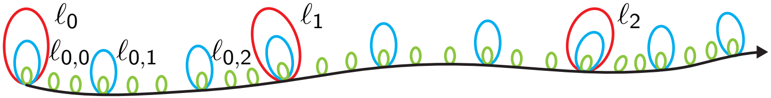

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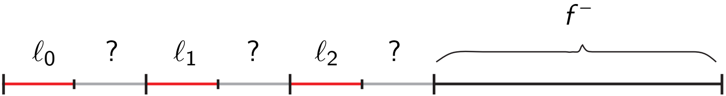

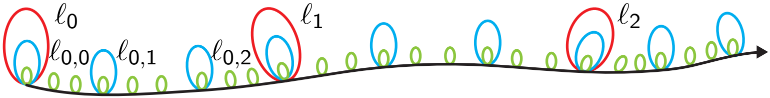

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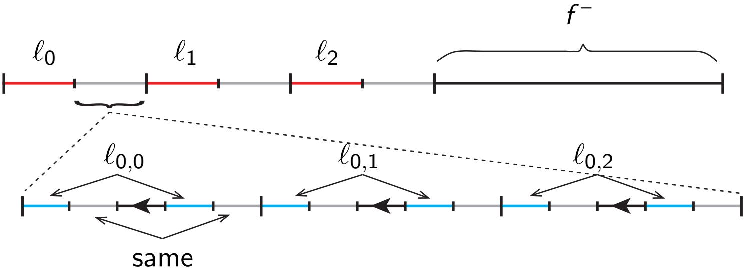

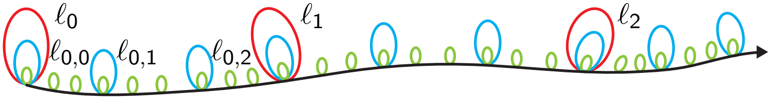

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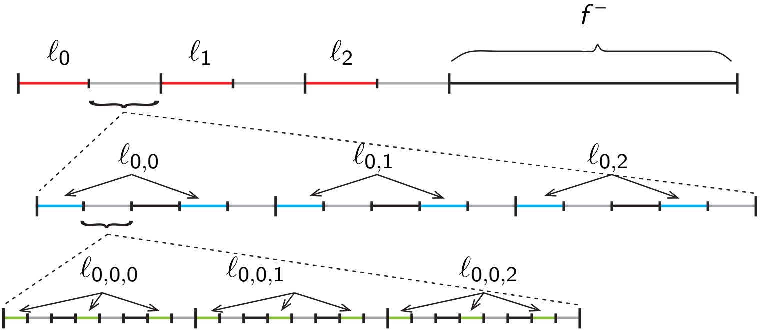

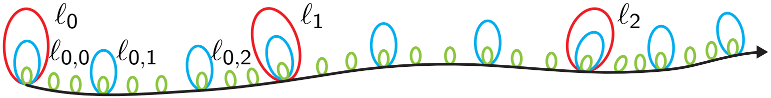

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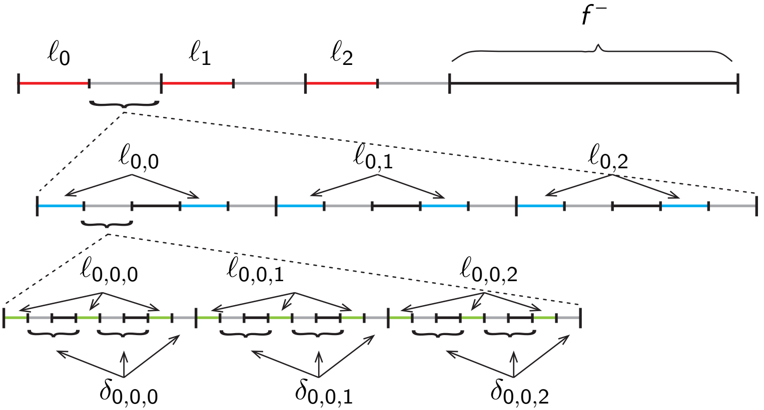

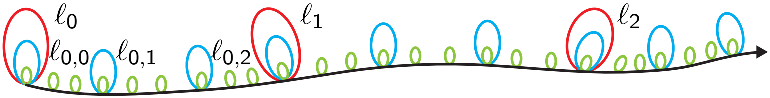

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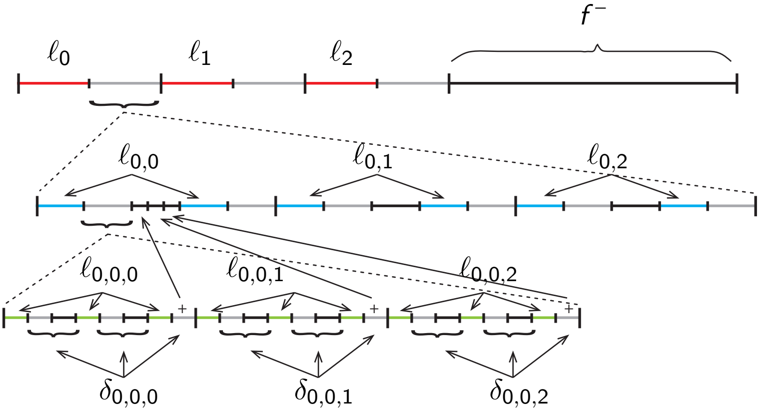

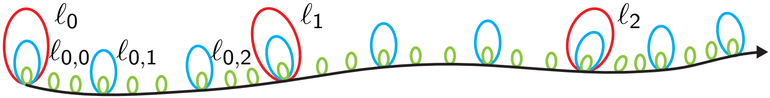

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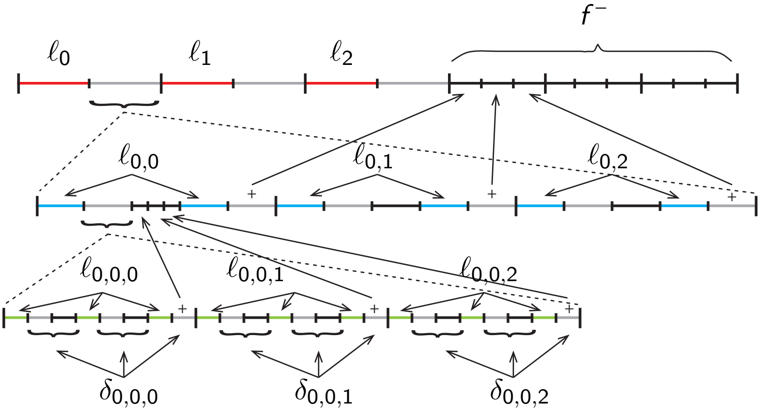

Rearrange the paths in $H_1(X)$ so that

$$[\ell] = \sum_{i=1}^{n-1} i! \sum_{s \in S_i} [\ell_s] + n! \left(\sum_{s \in S_n} [\ell_s] + [\delta_s] \right).$$

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Hence,

$$n! \mid h([\ell]) - \sum_{i=1}^{n-1} i! u_i a \qquad \text{in } A \text{ for all } n \geq 2.$$

□

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So, $h = \phi \circ \mu : \pi_1(X, *) \to A$ with $\bigcap_{\mathcal{U}} h(\pi(\mathcal{U}, x)) = \mathbb{J}_p \neq \{0\}$. $\square$

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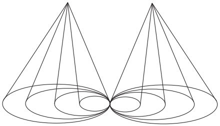

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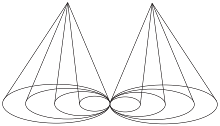

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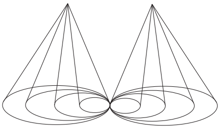

$$\pi^s(\mathbb{G},*) = \pi_1(\mathbb{G},*) \twoheadrightarrow H_1(\mathbb{G}) \twoheadrightarrow \mathbb{Q}, \quad H_1(\mathbb{G}) \twoheadrightarrow \mathbb{J}_p, \quad H_1(\mathbb{G}) \twoheadrightarrow \mathbb{Z}/p\mathbb{Z}$$

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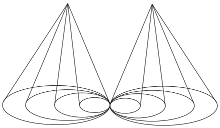

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Theorem

$$H_1(\mathbb{G}) = \prod_{\mathbb{N}} \mathbb{Z} / \bigoplus_{\mathbb{N}} \mathbb{Z} = \left(\bigoplus_{2^{\aleph_0}} \mathbb{Q} \right) \oplus \left(\prod_{p \in \mathbb{P}} A_p \right)$$

where $A_p = \prod_{\aleph_0} \mathbb{J}_p =$ p-adic completion of $\bigoplus_{2^{\aleph_0}} \mathbb{J}_p$

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Then $f = f_1 f_2 \cdots f_n$ such that $g = [f_1][f_2]\cdots[f_n]$ and, for each i, either $f_i \subseteq \mathbb{H}_1$, or $f_i \subseteq \mathbb{H}_2$, or f_i is paired with one other $f_j = f_i^-$.

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Suppose: $a^m = b^n c$ for some $b \in \pi_1(\mathbb{H})$, $c \in N$, $m \geqslant 1$, $n \geqslant 0$.

Show: $n > 0$ and $n \mid m$.

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Hence $n \mid 1$.

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For $\alpha = (s_k)_k \in \{1,2\}^{\mathbb{N}}$, put $N_\alpha = \{ \sum_{k=1}^{n} s_k 2^{n-k} \mid n \in \mathbb{N} \} \subseteq \mathbb{N}$.

Then N_α is infinite and $N_\alpha \cap N_\beta$ is finite $\forall \alpha \neq \beta$.

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For $N_\alpha = \{k_1 < k_2 < \cdots\}$ put $a_\alpha = [\ell_{2k_1-1} \ell_{2k_1} \ell_{2k_2-1} \ell_{2k_2} \ell_{2k_3-1} \ell_{2k_3} \cdots]$.

Then $\bigoplus_{2^{\aleph_0}} \mathbb{Z} \cong \langle a_\alpha N \mid \alpha \in \{1,2\}^{\mathbb{N}} \rangle$ is pure in $H_1(\mathbb{G}) = \pi_1(\mathbb{H})/N$.