Topological Ramsey spaces in creature forcing

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1. Forcings $\mathbb{P}_\alpha$ ($\alpha < \omega_1$) of Laflamme in [D/Todorcevic 2014,15 TAMS];
2. Forcings of Baumgartner and Taylor, of Blass, and others in [D/Mijares/Trujillo AFML];
3. $\mathcal{P}(\omega^\alpha)/\mathrm{Fin}^{\otimes\alpha}$, $2 \leq \alpha < \omega_1$ in [D 2015 JSL, 2016 JML].

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Moreover, the forced ultrafilters have *complete combinatorics* over $L(\mathbb{R})$ in the presence of a supercompact cardinal [Di Prisco/Mijares/Nieto].

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Thm. [R/S] Under certain hypotheses on a creature forcing, given a pure candidate $\bar{t}$ and a coloring $c : \mathrm{pos}(\bar{t}) \to 2$ there is a pure candidate $\bar{s}$ stronger than $\bar{t}$ such that c is constant on $\mathrm{pos}(\bar{s})$.

Cor. [R/S] (CH) There is an ultrafilter $\mathcal{U}$ on base set $\mathcal{F}_{\mathbf{H}}$ generated by $\{\text{pos}(\bar{t}_\alpha) : \alpha < \omega_1\}$ for a decreasing sequence of pure candidates $\langle \bar{t}_\alpha : \alpha < \omega_1 \rangle$, moreover, satisfying the previous partition theorem: For any $\bar{t}$ such that $\text{pos}(\bar{t}) \in \mathcal{U}$ and any partition of $\text{pos}(\bar{t})$ into finitely many pieces, there is a pure candidate $\bar{s} \leq \bar{t}$ such that $\text{pos}(\bar{s})$ is contained in one piece of the partition and $\text{pos}(\bar{s}) \in \mathcal{U}$.

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Remark. This is similar to the construction of an ultrafilter $\mathcal{U}$ on base set FIN generated by block sequences and using Hindman's Theorem so that for each partition of FIN into finitely many pieces, there is an infinite block sequence X such that $[X]$ is contained in one piece of the partition and $[X] \in \mathcal{U}$.

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Remark. The proofs in [R/S] use the Galvin-Glazer method extended to certain classes of creature forcings.

We now look at a specific example of a creature forcing in [R/S 2013].

Example 2.10 in [Roslanowski/Shelah 2013]

$\mathbf{H}_1(n) = n + 1$, for each $n < \omega$.

$\mathcal{F}_{\mathbf{H}_1} = \{\text{functions } f : \text{dom}(f) \text{ is finite and } \forall n \in \text{dom}(f)(f(n) \leq n)\}$.

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- $\mathbf{nor}[t] = \log_2(|A^t|)$,
- $\mathbf{val}[t] \subseteq \prod_{j \in u} \mathbf{H}_1(j) = \prod_{j \in u}(j + 1)$ s.t. $\{f(i^t) : f \in \mathbf{val}[t]\} = A^t$.

The Sub-Composition Operation: For $t_0, \dots, t_n \in K_1$ with $m^{t_l}_{\mathrm{up}} = m^{t_{l+1}}_{\mathrm{dn}}$ for all $l \le n$,

$\Sigma^*_1(t_0, \dots, t_n)$ is all $t \in K_1$ such that $m^t_{\mathrm{dn}} = m^{t_0}_{\mathrm{dn}}$, $m^t_{\mathrm{up}} = m^{t_n}_{\mathrm{up}}$, and

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$\mathrm{PC}^{\mathtt{tt}}_\infty(K_1, \Sigma^*_1)$ denotes the set of all **pure candidates** $\bar{t} = (t_0, t_1, \dots)$ such that for each $n < \omega$, $t_n \in K_1$ and $m^{t_n}_{\mathrm{up}} = m^{t_{n+1}}_{\mathrm{dn}}$, and $\lim_{n \to \infty} \mathbf{nor}[t_n] = \infty$.

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$\bar{s} \le \bar{t}$ iff $\exists\ (n_j)_{j<\omega}$ strictly increasing s.t. $\forall j$, $s_j \in \Sigma^*_1(t_{n_j}, \dots, t_{n_{j+1}-1})$.

The set of *possibilities* on the pure candidate $\bar{t}$ is

$$\mathrm{pos}^{\mathtt{tt}}(\bar{t}) = \bigcup\{f_0 \cup \cdots \cup f_n : n \in \omega \wedge \forall i \leq n\,(f_i \in \mathbf{val}[t_i])\}. \qquad (1)$$

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Thm. [R/S] Given $\bar{t} \in \mathsf{PC}^{\mathtt{tt}}_{\infty}(K_1, \Sigma^*_1)$, $l \geq 1$, $d_k : \mathrm{pos}^{\mathtt{tt}}(\bar{t} \upharpoonright k) \to l$, $k < \omega$, $\exists \bar{s} \leq \bar{t}$ in $\mathsf{PC}^{\mathtt{tt}}_{\infty}(K_1, \Sigma^*_1)$ and a $l' < l$ such that for each $i < \omega$, if k is such that $s_i \in \Sigma^*_1(\bar{t} \upharpoonright k)$ and $f \in \mathrm{pos}^{\mathtt{tt}}(\bar{s} \upharpoonright i)$, then $d_k(f) = l'$.

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Thm. [R/S] Given $\bar{t} \in \mathsf{PC}^{\mathtt{tt}}_{\infty}(K_1, \Sigma_1^*)$, $l \geq 1$, $d_k : \mathrm{pos}^{\mathtt{tt}}(\bar{t} \upharpoonright k) \to l$, $k < \omega$, $\exists \bar{s} \leq \bar{t}$ in $\mathsf{PC}^{\mathtt{tt}}_{\infty}(K_1, \Sigma_1^*)$ and a $l' < l$ such that for each $i < \omega$, if k is such that $s_i \in \Sigma_1^*(\bar{t} \upharpoonright k)$ and $f \in \mathrm{pos}^{\mathtt{tt}}(\bar{s} \upharpoonright i)$, then $d_k(f) = l'$.

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Remark. This theorem will be recovered from showing that there is a topological Ramsey space dense in $\mathsf{PC}^{\mathtt{tt}}_{\infty}(K_1, \Sigma_1^*)$.

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What is a topological Ramsey space?

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Def. $\mathcal{X} \subseteq [\omega]^\omega$ is **Ramsey** iff for each $[a, X]$, there is $a \sqsubset Y \subseteq X$ such that either $[a, Y] \subseteq \mathcal{X}$ or $[a, Y] \cap \mathcal{X} = \emptyset$.

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Thm. [Ellentuck 1974] Every $\mathcal{X} \subseteq [\omega]^\omega$ with the property of Baire is Ramsey, and every meager set is Ramsey null.

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This extends theorems of Nash-Williams, Galvin and Prikry, and Silver.

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Def. [Todorcevic] A triple $(\mathcal{R}, \leq, r)$ is a **topological Ramsey space** if every subset of $\mathcal{R}$ with the Baire property is Ramsey, and every meager subset of $\mathcal{R}$ is Ramsey null.

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Abstract Ellentuck Thm. [Todorcevic] If $(\mathcal{R}, \leq, r)$ satisfies Axioms **A.1 - A.4** and $\mathcal{R}$ is closed (in $\mathcal{AR}^{\mathbb{N}}$), then $(\mathcal{R}, \leq, r)$ is a topological Ramsey space.

Selective Coideals and Complete Combinatorics

Given a topological Ramsey space $(\mathcal{R}, \le, r)$, a coideal $\mathcal{U} \subseteq \mathcal{R}$ is **selective** if for each $A \in \mathcal{U}$ and any collection $(A_a)_{a \in \mathcal{AR}|A}$ of members of $\mathcal{U} \restriction A$, there is a $U \in \mathcal{U}$ which diagonalizes $(A_a)_{a \in \mathcal{AR}|A}$.

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To each topological Ramsey space there corresponds a notion of **almost reduction** $\le^*$, and forcing with $(\mathcal{R}, \le^*)$ adds a selective coideal $\mathcal{U}$ on $\mathcal{R}$.

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Thm. [DiPrisco/Mijares/Nieto] In the presence of a supercompact cardinal, every selective coideal $\mathcal{U} \subseteq \mathcal{R}$ is generic for $(\mathcal{R}, \le^*)$.

A dense subset of $\mathrm{PC}^{\mathtt{tt}}_{\infty}(K_1, \Sigma_1^*)$ forming a tRs

Recall: $\mathbf{H}_1(n) = n + 1$.

Creatures $t \in K_1$ are determined by $m^t_{\mathrm{dn}} < m^t_{\mathrm{up}}$, $u^t \subseteq [m^t_{\mathrm{dn}}, m^t_{\mathrm{up}})$, $i^t \in u^t$, $A^t \subseteq \mathbf{H}_1(i^t)$, $\mathbf{val}[t] \subseteq \prod_{j \in u^t} \mathbf{H}_1(j)$ satisfying $\{f(i^t) : f \in \mathbf{val}[t]\} = A^t$.

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$\mathcal{R}(K_1, \Sigma_1)$ is the set of $\bar{t} = (t_n : n < \omega) \in \mathrm{PC}^{\mathtt{tt}}_{\infty}(K_1, \Sigma^*_1)$ such that $\forall n$,

1. $|A^{t_n}| = n + 1$ and
2. for each $a \in A^{t_n}$, there is exactly one function $g^{t_n}_a \in \mathbf{val}[t_n]$ such that $g_a(i^{t_n}) = a$.

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2. for each $a \in A^{t_n}$, there is exactly one function $g_a^{t_n} \in \mathbf{val}[t_n]$ such that $g_a(i^{t_n}) = a$.

Thus, $\mathbf{val}[t_n] = \{g_a^{t_n} : a \in A^{t_n}\}$ and $|\mathbf{val}[t_n]| = |A^{t_n}| = n + 1$.

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Using the fact that for $\bar{t} \in \mathcal{R}(K_1, \Sigma_1^*)$, $|\mathrm{pos}^{\mathtt{tt}}(t_n)| = n + 1$ for each n, we can quickly derive Rosłanowski and Shelah's result for this example, and hence obtain an ultrafilter on $\mathcal{F}_{\mathbf{H}_1}$ which satisfies the partition theorem of [R/S].

The proof that $(\mathcal{R}(K_1, \Sigma_1^*), \leq, r)$ is a topological Ramsey space hinges on proving the pigeonhole principle **Axiom A.4**:

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Members $(t_0, \dots, t_{k-2}, x)$ of $r_k[k-1, \bar{t}\,]$ are completely determined by the triple $(i^x, A^x, m^x_{\mathrm{up}})$. So c is really coloring

$$\bigcup_{n \geq k-1} \bigcup_{k-1 \leq p \leq n} A^{t_{k-1}} \times \cdots \times A^{t_{p-1}} \times [A^{t_p}]^k \times A^{t_{p+1}} \times A^{t_n}.$$

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This looks suspiciously similar to the following theorem.

Thm. [DiPrisco/Llopis/Todorcevic 2004] There is an $R : (\mathbb{N}^+)^{<\omega} \to \mathbb{N}^+$ such that for every infinite sequence $(m_j)_{j<\omega}$ of positive integers and for every coloring

$$c : \bigcup_{n<\omega} \prod_{j\le n} R(m_0, \ldots, m_j) \to 2,$$

there exist $H_j \subseteq R(m_0, \ldots, m_j)$, $|H_j| = m_j$, for $j < \omega$, such that c is constant on the product

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Remark. The difference is that we need sets of size k to be able to move up and down indices of the product.

As an intermediate step to the new product tree Ramsey theorem we prove

Thm. [D] Given $k \geq 1$, there is a function $R_k : [\mathbb{N}^+]^{<\omega} \to \mathbb{N}^+$ such that for each sequence $(m_j)_{j<\omega}$ of positive integers, for each coloring

$$c : \bigcup_{n<\omega} [R_k(m_0)]^k \times \prod_{j=1}^{n} R_k(m_0, \dots, m_j) \to 2,$$

there are subsets $H_j \subseteq R_k(m_0, \dots, m_j)$ such that $|H_j| = m_j$ and c is constant on

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Then diagonalize and apply Theorem [DLT] to obtain the next theorem.

New Product Tree Ramsey Theorem

For $p \leq n$, $[K_p]^k \times \prod_{j \in (n+1)\setminus\{p\}} K_j$ denotes

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Thm. [D] Given $k \ge 1$, a sequence of positive integers $(m_0, m_1, \dots)$, sets K_j, $j < \omega$ such that $|K_j| \ge j+1$, and a coloring

$$c : \bigcup_{n<\omega} \bigcup_{p\le n} ([K_p]^k \times \prod_{j\in(n+1)\setminus\{p\}} K_j) \to 2,$$

there are infinite sets $L, N \subseteq \omega$ such that, enumerating L and N in increasing order, $l_0 \le n_0 < l_1 \le n_1 < \dots$, and there are subsets $H_j \subseteq K_j$, $j < \omega$, such that $|H_{l_i}| = m_i$ for each $i < \omega$, $|H_j| = 1$ for each $j \in \omega \setminus L$, and c is constant on

$$\bigcup_{n\in N} \bigcup_{l\in L\cap(n+1)} ([H_l]^k \times \prod_{j\in(n+1)\setminus\{l\}} H_j).$$

Example 2.11 in [Roslanowski/Shelah 2013]

$\mathbf{H}_2(n) = 2$ for $n < \omega$. $\mathcal{F}_{\mathbf{H}_2} = \{f : \text{dom}(f) \in \text{FIN} \text{ and } f : \text{dom}(f) \to 2\}$.

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K_2 = set of all creatures $t = (\mathbf{nor}[t], \mathbf{val}[t], \mathbf{dis}[t], m^t_{\mathrm{dn}}, m^t_{\mathrm{up}})$ such that

- $\emptyset \neq \mathbf{dis}[t] \subseteq [m^t_{\mathrm{dn}}, m^t_{\mathrm{up}})$,
- $\mathbf{val}[t] \subseteq {}^{\mathbf{dis}[t]}2$,
- $\mathbf{nor}[t] = \log_2(|\,\mathbf{val}[t]|)$.

For $t_0, \dots, t_n \in K_2$ with $m^{t_l}_{\mathrm{up}} \leq m^{t_{l+1}}_{\mathrm{dn}}$ for all $l \leq n$, $\Sigma_2(t_0, \dots, t_n)$ consists of all creatures $t \in K_2$ such that

$$m^t_{\mathrm{dn}} = m^{t_0}_{\mathrm{dn}},\ m^t_{\mathrm{up}} = m^{t_n}_{\mathrm{up}},\ \mathbf{dis}[t] = \mathbf{dis}[t_l], \mathbf{val}[t] \subseteq \mathbf{val}[t_l], \text{for some } l \leq n.$$

$\mathsf{PC}_\infty(K_2, \Sigma_2)$ denotes the set of all **pure candidates** $\bar{t} = (t_0, t_1, \dots)$ such that for each $i < \omega$, $t_i \in K_2$ and $m^{t_i}_{\mathrm{up}} \leq m^{t_i}_{\mathrm{dn}}$, and $\lim_{i\to\infty} \mathbf{nor}[t_i] = \infty$.

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$\bar{s} \leq \bar{t}$ iff $\exists$ $(j_n)_{n<\omega}$ strictly increasing s.t. $\forall n$, $s_n \in \Sigma_2(t_{j_{2n}}, \dots, t_{j_{2n+1}})$.

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Thm. [R/S] Given $\bar{t} \in \mathsf{PC}_\infty(K_2, \Sigma_2)$, $l \ge 1$, and $d_k : \mathrm{pos}(\bar{t} \restriction k) \to l$, $k < \omega$, there exist $\bar{s} \le \bar{t}$ in $\mathsf{PC}_\infty(K_2, \Sigma_2)$ and $l' < l$ such that for each $i < \omega$, if k is such that $s_i \in \Sigma_2(\bar{t} \restriction k)$ and $f \in \mathrm{pos}(\bar{s} \restriction i)$, then $d_k(f) = l'$.

This theorem will be recovered by showing that there is a topological Ramsey space dense in $\mathsf{PC}_\infty(K_2, \Sigma_2)$.

A dense subsets forming a topological Ramsey space

$\mathcal{R}(K_2, \Sigma_2) = \{\bar{s} \in \mathrm{PC}_\infty(K_2, \Sigma_2) : \forall l < \omega, \ |\,\mathbf{val}[t_l]\,| = l + 1\}$,

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Remark. The proof of the pigeonhole again relies on the new product tree Ramsey theorem. The application, though, is slightly different.

The generic filter

Since $\mathcal{R}(K_2, \Sigma_2)$ is a topological Ramsey space, it forces a generic filter $\mathcal{G}$ which is selective for $\mathcal{R}(K_2, \Sigma_2)$, hence has complete combinatorics over $L(\mathbb{R})$ in the presence of a supercompact cardinal.

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The generic filter induces an ultrafilter $\mathcal{U}$ on $\mathcal{AR}_1$.

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This filter induces an ultrafilter on $\mathcal{F}_{\mathbf{H}_2} = \{f : \mathrm{dom}(f) \in \mathrm{FIN}$ and $f : \mathrm{dom}(f) \to 2\}$ generated by possibilities on pure candidates and satisfying the partition theorem of [R/S].

Example 2.13 in [Rosłanowski/Shelah 2013]

Let $N > 0$ and $\mathbf{H}_N(n) = N$ for $n < \omega$. K_N consists of all creatures t s.t.

- $\mathbf{dis}[t] = (X_t, \varphi_t)$, where $X_t \subsetneq [m^t_{\mathrm{dn}}, m^t_{\mathrm{up}})$, and $\varphi_t : X_t \to N$,
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- for each $l \leq n$, either $X_t \cap [m^{t_l}_{\mathrm{dn}}, m^{t_l}_{\mathrm{up}}) = X_{t_l}$ and $\varphi_t \restriction [m^{t_l}_{\mathrm{dn}}, m^{t_l}_{\mathrm{up}}) = \varphi_{t_l}$,
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Thm. [R/S] Given a pure candidate $\bar{t}$ and a coloring $c : \mathrm{pos}^{\mathtt{tt}}(\bar{t}) \rightarrow 2$, there is an $\bar{s} \leq \bar{t}$ such that c is constant on $\mathrm{pos}^{\mathtt{tt}}(\bar{s})$.

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Remark. This space is the *tight* version of the Carlson-Simpson space of variable words.

Questions.

1. What other creature forcings are essentially (topological) Ramsey spaces? Extend this study to streamline approaches to certain classes of creature forcings.
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Thank you for your attention.

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