Accessible arc-components and extendability of the shift homeomorphism of planar embeddings of unimodal inverse limit spaces

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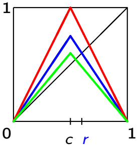
Introduction

Let I denote a unit interval and let $T : I \to I$ be a unimodal map such that $T(0) = 0$.



Let c denote the *critical point* of map T. We say $[T^2(c), T(c)]$ is the *core* of T.

Note that T has two fixed points: 0 and r.

Inverse limit spaces

We define the *inverse limit space* $\varprojlim(I, T)$ *with a bonding map* T by

$$\varprojlim(I, T) := \{x = (\ldots, x_2, x_1, x_0) \in I^\infty; T(x_{(n-1)}) = x_n, \forall\, n \in \mathbb{N}\},$$

equipped with a metric

$$d(x, y) = \sum_{i \geq 0} \frac{|x_i - y_i|}{2^i}$$

for every $x, y \in \varprojlim(I, T)$ and the *shift homeomorphism* $\sigma : \varprojlim(I, T) \to \varprojlim(I, T)$, defined by

$$\sigma((\ldots, x_2, x_1, x_0)) = (\ldots, x_1, x_0, T(x_0)).$$

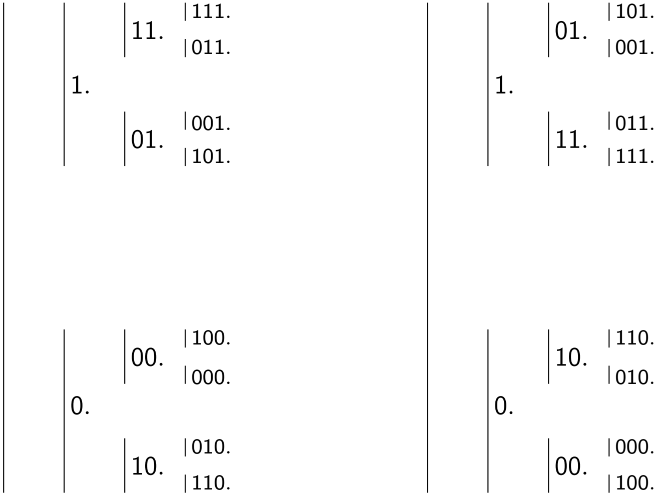
Embeddings making an arbitrary point accessible

Thm (Anušić, Bruin, Č., 2016): For an arbitrary point $x \in \varprojlim(I, T)$ there exists an embedding of $\varprojlim(I, T)$ which makes x accessible.

Idea of the proof

- prescribe a two-sided infinite itinerary to every point $x \in \varprojlim(I, T)$ where the left infinite itinerary $\overleftarrow{x}$ determines the basic arc $x \in A(\overleftarrow{x})$,
- determine the rule on admissible left-infinite sequences for which the left infinite code $L := \dots l_n \dots l_1. \in \{0,1\}^{-\mathbb{N}}$ of $A(\overleftarrow{x})$ is the largest code among basic arcs,
- align $A(\overleftarrow{x})$ as horizontal arcs along the vertically embedded Cantor set in the plane.

Coding the Cantor set



(a) (b)

Figure: Coding the Cantor set with respect to (a) $L = \ldots 111.$ and (b) $L = \ldots 101.$

Example of a constructed embedding

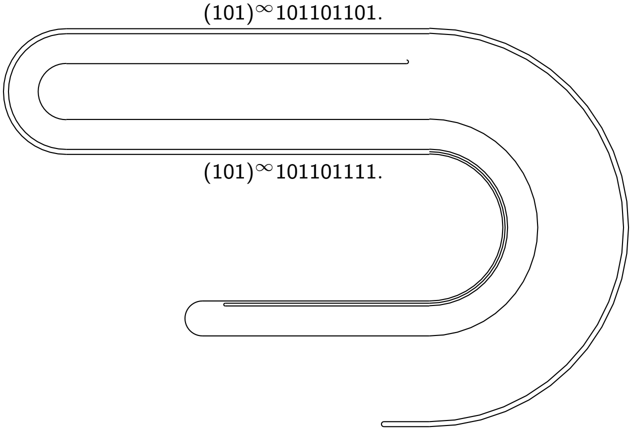


Figure: The planar representation of an arc $A \subset \varprojlim(I, T)$

Denote all of the constructed embeddings varying L by $\mathcal{E}$.

Preliminaries

Def: The arc-component $\mathcal{U}(x)$ of a point $x \in K$ in a continuum K is a union of all arcs in K containing a point x.

Remark: Possible arc-components of a point x of a chainable continuum:

- the point x,
- an arc containing x,
- a line (continuous image of $\mathbb{R}$) containing x,
- a ray (continuous image of $\mathbb{R}^+$) containing x.

- $(\ldots 0, 0, 0) \in \mathcal{C} \subset \varprojlim(T, I)$,

- $(\ldots r, r, r) \in \mathcal{R} \subset \varprojlim(T, I)$.

Motivation

Note that $\varprojlim(I,T) = \mathcal{C} \cup \varprojlim([T^2(c), T(c)], T)$ (Bennet, 1962) and we are interested in spaces where $\varprojlim([T^2(c), T(c)], T)$ (the core inverse limit) is indecomposable.

Def: A point $x \in X \subset \mathbb{R}^2$ is *accessible* if there exist an arc $A = [a, b]$ such that $a = x$ and $A \cap X = \{x\}$. An arc-component is fully accessible, if every $x \in \mathcal{U}(x)$ is accessible.

- K. Brucks, B. Diamond (1995): Embeddings of $\varprojlim(I,T)$ making $L = 0^{-\infty}1$. the largest basic arc.
- H. Bruin (1999): Embedding of $\varprojlim(I,T)$ such that every point in $\mathcal{R}$ ($L = 1^{-\infty}$.) is fully accessible, extending σ homeomorphism to the plane.

We call these two embeddings of $\varprojlim(I,T)$ standard.

Extendability of standard embeddings

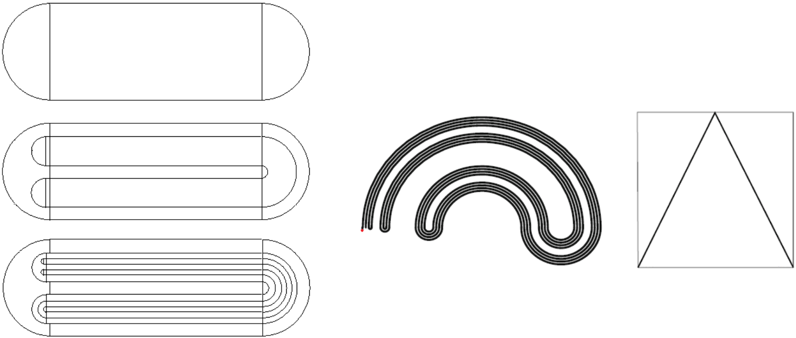


Figure: Smale's horseshoe

Extendability of σ to the plane

Question (Boyland, 2015): Do there exist embeddings of $\varprojlim(I, T)$ which are not equivalent to standard embeddings and σ-homeomorphism is extendable to the plane?

Accessible points of embeddings $\mathcal{E}$

Question: What are the accessible points of embeddings $\mathcal{E}$?

Observe itineraries of points through finite cylinders $[a_1 \dots a_n]$ for some $a_i \in \{0, 1\}$!

Remark: If $A(\overleftarrow{x})$ is on the top/bottom of some finite cylinder $[a_1 \dots a_n]$, then every point $x \in A(\overleftarrow{x})$ is accessible.

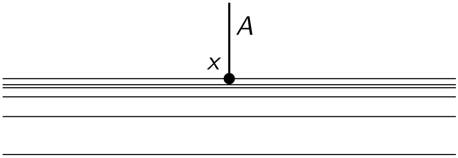
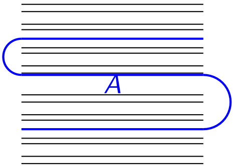


Figure: Point on the top of some cylinder is accessible.

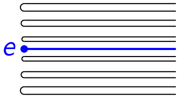
Sets of accessible points of $\mathcal{E}$

- There exist embeddings from $\mathcal{E}$ where an arc-component is partially accessible.

- Even countably many arc-components are partially accessible in some embeddings $\mathcal{E}$!



- For some embeddings from $\mathcal{E}$, endpoint e where $\mathcal{U}(\overleftarrow{e}) \neq \mathcal{U}(A(L))$ is accessible but every $e \neq x \in \mathcal{U}(e)$ is not.



- The ray $(\dots 0, 0) \in \mathcal{C}$ is fully accessible in every embedding $\mathcal{E}$ except for $\varprojlim(I, T)$ being Knaster continuum.

- Let $L = 1^{-\infty}$.. Arc-component $\mathcal{R}$ is fully accessible and every point $x \notin \mathcal{R}$ is not.

- Kneading sequence starting with $\nu = 101\dots$ and $L = (01)^{-\infty}$. Two arc-components from $\varprojlim([T^2(c), T(c)], T)$ coded by $(01)^{-\infty}$. and $(10)^{-\infty}$. are fully accessible!

- Arc-component $\mathcal{U}(A(L))$ is fully accessible for every embedding $\mathcal{E}$.

The number of fully accessible arc-components

Question: Does there exist an embedding of an indecomposable chainable continuum in the plane so that more than 2 different nondegenerate arc-components are fully accessible?

Yes!

Embeddings of $\varprojlim([T^2(c), T(c)], T)$ making $n \in N$ arc-components fully accessible

Thm (Anušić, Bruin, Č., 2016): Let $\varprojlim(I, T)$ have $\nu = (10\dots01)^\infty$ periodic with period κ. For the embedding of $\varprojlim([T^2(c), T(c)], T)$ making $L = 0^{-\infty}1$ the largest sequence, all κ arc-components with left-infinite itinerary $(10\dots01)^\infty x_n \dots x_1$. for some $n \in \mathbb{N}$ are fully accessible.

Sketch of a proof

- Embedding with $L = 0^{-\infty}1$ is exactly Brucks & Diamond embedding, homeomorphism σ can be extended from $\varprojlim(I, T)$ to the plane.

- There exists $H : \mathbb{R}^2 \to \mathbb{R}^2$ planar homeomorphism with $H|_{\mathcal{C} \cup \varprojlim([T^2(c), T(c)], T)} = \sigma$ and thus $H|_{\varprojlim([T^2(c), T(c)], T)} = \sigma$.

- σ permutes endpoints $e_0, \ldots, e_{\kappa-1} \in \varprojlim([T^2(c), T(c)], T)$ and arc-components $\mathcal{U}(e_0), \ldots \mathcal{U}(e_{\kappa-1})$.

Sketch of a proof

- Symbolic arguments give that all basic arcs from $\mathcal{U}(e_0), \dots \mathcal{U}(e_{\kappa-1})$ are tops/bottoms of cylinders and no other basic arcs are top/bottom of some cylinder.
- for $\varprojlim([T^2(c), T(c)], T)$ map σ is extendable to the plane and thus $\mathcal{U}(e_k)$ accessible for every $k \in \{0, \dots \kappa - 1\}$.

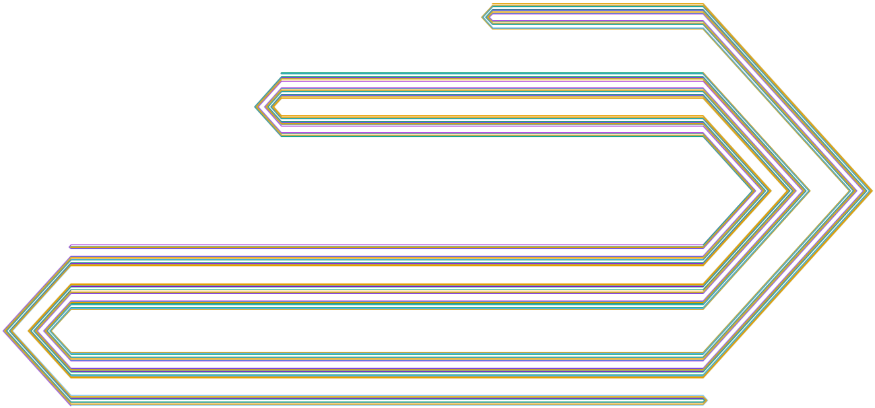


Figure: $\nu = (1001)^\infty$, $\mathcal{U}(e_0)$, $\mathcal{U}(e_1)$, $\mathcal{U}(e_2)$, $\mathcal{U}(e_3)$

Corollary (Anušić, Bruin, Č., 2016): For every $n \in \mathbb{N}$ there exists an indecomposable continuum with n different fully accessible non-degenerate arc-components.

Question: Does there exist an embedding of an indecomposable chainable continuum in the plane so that countably many non-degenerate arc-components are fully accessible?

Non-extendability of σ to the plane of embeddings $\mathcal{E}$

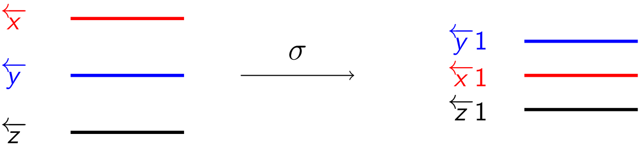
Prop: Fix $\varprojlim([T^2(c), T(c)], T)$ and $L = \dots l_n \dots l_1.$ such that $\mathcal{U}(A(L)) \neq \mathcal{R}, \mathcal{C}$. For embedding making L the largest sequence, σ homeomorphism is not extendable to the plane.

Idea of the proof:

- Assume σ is extendable,
- $\mathcal{U}(A(L))$ is always fully accessible,
- if $L \neq (01)^{-\infty}.$ and $\nu \neq 101\dots$ there exists $k \in \mathbb{N}$ such that $\mathcal{U}(\sigma^k(A(L)))$ is not accessible.

$L = (01)^{-\infty}$ and $\nu = 101\ldots$

Exactly 2 fully accessible arc-components with left infinite itinerary $(01)^{-\infty}. = \sigma((10)^{-\infty}.)$ and $(10)^{-\infty}.$ and no other point from $\varprojlim([T^2(c), T(c)], T)$ is accessible.



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Thank you!