Ultrafilter-completeness on a zero-sets of uniformly continuous functions

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Introduction

It is known, that an epi-reflective hull $\mathfrak{L}([0,1])$ of the unit segment $I = [0,1]$ in a category *Tych* consists of all closed subspaces of powers of $[0,1]$. Stone–Čech compactification βX of Tychonoff space X is a projective object in $\mathfrak{L}([0,1])$, i.e. βX is the essentially unique compactum containing X densely such that each continuous mapping $f : X \to K$ $(K \in \mathfrak{L}([0,1]))$ admits a continuous extension $\beta f : \beta X \to K$, or $\beta : X \to \beta X$ is an epi-reflection and homeomorphic embedding [Gillman–Jerison, 1960; Walker, 1974; Engelking, 1989]. The unique uniformity of compactum βX induces on X Stone-Čech uniformity u_β, whose base consists of all finite cozero coverings (cozero covering consists of cozero sets). The uniformity u_β is a precompact reflection [Isbell, 1964] of many uniformities on X (for example, Nachbin uniformity or Shirota uniformity) and among them there is a maximal uniformity u_f being a fine uniformity, whose base consists of all locally finite cozero coverings [Gillman–Jerison, 1960; Engelking, 1989].

For Tychonoff space X zero-sets $\mathcal{Z}(X)$ of all continuous functions form *separating, nest-generated intersection ring* (s.n.–g.i.r.)[Steiner A. K., Steiner E.F., 1970] and Wallman compactification $\omega(X, \mathcal{Z}(X))$ is Stone-Čech compactification βX [Gillman–Jerison, 1960]. An elements of βX are all $z-$ultrafilteres ($\equiv$ maximal centered systems of $\mathcal{Z}(X)$). All countably centered $z-$ultrafilteres part of βX forms Hewitt extension υX and another part of βX of all locally finite additive $z-$ultrafilteres forms Dieudonne completion μX [Gillman–Jerison, 1960; Curzer–Hager, 1976] and υX is a projective object in the epi-reflective hull $\mathfrak{L}(\mathbb{R})$ ($\equiv$ all closed subspaces of powers of $\mathbb{R}$), i.e. υX is the essentially unique realcompact space containing X densely such that each continuous mapping $f : X \to Y$ ($Y \in \mathfrak{L}(\mathbb{R})$) admits a continuous extension $\upsilon f : \upsilon X \to Y$, or $\upsilon : X \to \upsilon X$ is an epi-reflection and homeomorphic embedding, μX is a projective object in the epi-reflective hull $\mathfrak{L}(\mathcal{M})$ ($\equiv$ all closed subspaces of products from a class $\mathcal{M}$), where $\mathcal{M}$ is a class of all metric spaces [Franklin, 1971; Herrlich, 1971; Hager, 1975], i.e. μX is the essentially unique Dieudonne complete space containing X densely such that each continuous mapping $f : X \to Y$ ($Y \in \mathfrak{L}(\mathcal{M})$) admits a continuous extension $\mu f : \mu X \to Y$, or $\mu : X \to \mu X$ is an epi-reflection and homeomorphic embedding.

Samuel compactification s_uX of a uniform space uX is a projective object in the epi-reflective hull $\mathfrak{L}([0,1])$ in a category $Unif$, i.e. s_uX is the essentially unique compactum containing X densely such that each uniformly continuous mapping $f : uX \to K$ ($K \in \mathfrak{L}([0,1])$) admits a continuous extension $s_uf : s_uX \to K$, or it is an epi-reflection $s_u : uX \to s_uX$, at that it is not a uniform embedding [Isbell, 1964]. A compactum s_uX is the result of completion of X with respect to precompact reflection u_p of uniformity u (a base of u_p consists of all finite uniform coverings of uniformity u [Isbell, 1964]). It is known that there is not always the maximal uniformity for which u_p is its a precompact reflection [Ramm–Švarc, 1953].

Questions

Professor K.Kozlov asked: *What does uniformity correspond to* $\beta-$*like compactification of Tychonoff space in sense [Mrówka, 1973]? Does it exist a maximal uniformity, for which this precompact uniformity is a precompact reflection?*

Preliminaries

Any $\beta-$like compactification can be constructed as Wallman compactification by a base, which is s.n.–g.i.r. in sense [Steiner A. K., Steiner E.F., 1970]. For any uniform space uX zero-sets $\mathcal{Z}_u$ of all uniformly continuous functions form a normal base in sense [Frink, 1964] and Wallman compactification $\omega(X, \mathcal{Z}_u)$ [Frink, 1964; Aarts–Nishiura, 1993; Iliadis, 2005] is $\beta-$like compactification [Chekeev, 2016], which is denoted by $\beta_u X$. Hence $\mathcal{Z}_u$ is s.n.–g.i.r. A uniformity of compactum $\beta_u X$ induces on X precompact uniformity u_p^z, which is called *Wallman precompact uniformity*, and it has a base of all finite $u-$open coverings [Chekeev, 2016]. A maximal uniformity, for which u_p^z is precompact reflection, is a *coz*$-$fine uniformity u_{cf}^z in sense [Z.Frolik, 1975] and, we note, it has a base of all locally finite *coz*$-$additive $u-$open coverings. In this talk for uniform space uX a various kinds of completeness by z_u-ultrafilteres on $\mathcal{Z}_u$ are determined, corresponding to the well-known topological concepts, such as Stone-Čech compactification βX, Hewitt extension υX and Dieudonne completion μX.

For any uniform space uX $U(uX)$ ($U^*(uX)$) be a set of all (bounded) uniformly continuous functions, $\mathcal{Z}_u$ be a zero-sets of all functions of $U^*(uX)$ or $U(uX)$, $C\mathcal{Z}_u = \{X \setminus Z : Z \in \mathcal{Z}_u\}$ be a set of cozero-sets. Every set of $\mathcal{Z}_u$ ($C\mathcal{Z}_u$) is said to be $u-$*closed* ($u-$*open*) [Charalambous, 1975]
It is known, that:

Proposition 1.[M.G. Charalambous, 1975]

(1) $\mathcal{Z}_u$ is a base of closed set topology of a uniform space uX.

(2) $\mathcal{Z}_u$ is a normal base in sense [Frink, 1964].

(3) $C\mathcal{Z}_u$ is a base of open set topology of a uniform space uX.

Definition 2.[Z. Frolik, 1975; M.G. Charalambous, 1975, 1991]

A mapping $f : uX \to vY$ between uniform spaces is said to be a $coz-$*mapping*, if $f^{-1}(C\mathcal{Z}_v) \subseteq C\mathcal{Z}_u$ (or $f^{-1}(\mathcal{Z}_v) \subseteq \mathcal{Z}_u$) [Z. Frolik, 1975]. If $Y = \mathbb{R}$ or $Y = I$, then the $coz-$mapping $f : uX \to \mathbb{R}$ is said to be a $u-$*continuous function* and the $coz-$mapping $f : uX \to I$ is said to be a $u-$*function* [Charalambous, 1975, 1991].

We denote by $C_u(X)$ ($C_u^*(X)$) the set of all (bounded) $u-$continuous functions on a uniform space uX and it is known $C_u(X)$ forms an algebra with inversion [Chekeev, 2016] in sense [Hager-Johnson, 1968; Hager, 1969; Isbell, 1958].

Definition 3.

A maximal centered system of $u-$closed sets on a uniform space uX is said to be z_u-*ultrafilter*.

Below by means of z_u-ultrafilteres, satisfying additionally to the properties of being *countably centered* and *locally finite additivity* the concepts of z_u-*completeness*, $\mathbb{R}-z_u-$*completeness* and *weakly* z_u-*completeness* of a uniform spaces are introduced, their basic properties are established, which allow to obtain their characterizations in a category *ZUnif*, whose objects are uniform spaces, and morphisms are $coz-$mappings.

Let's introduce a concept of z_u-*completeness.*

Definition 4.

A uniform space uX is said to be z_u-*complete*, if every z_u-ultrafilter converges.

Proposition 5.

A uniform space uX is compact iff it is z_u-complete.

As it is above mentioned in Proposition 1, $\mathcal{Z}_u$ is a normal base and Wallman compactification $\omega(X, \mathcal{Z}_u)$ is $\beta-$like compactification in sense [Mrówka, 1973] and it has the next property is similar to Stone–Čech compactification.

Main Results

Theorem 6.

For every uniform space uX Wallman compactification $\omega(X, \mathcal{Z}_u) = \beta_u X$ is $\beta-$like compactification with the next equivalent properties:

(I) Every $coz-$mapping f from uX into any compactum K has a continuous extension $\beta_u f$ from $\beta_u X$ into K.

(II) uX is C_u^*-embedded in $\beta_u X$.

(III) Any two disjoint $u-$closed sets in uX have disjoint closures in $\beta_u X$.

(IV) For any two $u-$closed sets Z_1 and Z_2 in uX the equality $[Z_1 \cap Z_2]_{\beta_u X} = [Z_1]_{\beta_u X} \cap [Z_2]_{\beta_u X}$ holds.

(V) Distinct z_u-ultrafilters on uX have distinct limits in $\beta_u X$.

The compactification $\beta_u X$ is unique in the next sense: if a compactification Y of uX satisfies anyone of listed conditions, then there exists a homeomorphism of $\beta_u X$ onto Y that leaves X pointwise fixed.

Main Results

Under $C_u(C_u^*)-$*embedding* we will understand the next:

Definition 7.

Let X be a subspace of a Tychonoff space Y and u be a uniformity on X, v be a uniformity on Y such that $\mathcal{Z}_v \wedge X = \mathcal{Z}_u$. The uniform space uX is said to be $C_u(C_u^*)-$*embedded* in the uniform space vY, if any function of $C_u(X)$ ($C_u^*(X)$) can be extended to a function in $C_v(Y)$ ($C_v^*(Y)$).

Main Results

We introduce a concept of $\mathbb{R}-z_u-$*completeness*.
If p is z_u-ultrafilter and $\cap_{n\in\mathbb{N}} F_n \neq \emptyset$ for any subfamily $\{F_n\}_{n\in\mathbb{N}}$ of p, then p is said to be *countably centered* z_u-*ultrafilter*. We note that countably centered z_u-ultrafilter is *closed under countable intersections*.

Definition 8.

A uniform space uX is said to be $\mathbb{R}-z_u-$*complete*, if every countably centered z_u-ultrafilter converges.

Problem

Naturally a problem arises: *To characterize* $\mathbb{R}-z_u-$*complete uniform spaces*. This problem is connected with Wallman realcompactification in sense [Steiner A.K., Steiner E.F., 1970].

As it is known:

Proposition 9. [Steiner A.K.,Steiner E.F., 1970]

Wallman realcompactification $\upsilon(X, \mathcal{Z}_u) = \upsilon_u X$ of a uniform space uX is a subspace of $\beta_u X$ consisting of the set of all countably centered z_u-ultrafilteres on $\mathcal{Z}_u$.

Main Results

For Wallman realcompactification $\upsilon_u X$ of a uniform space uX the next characterizations hold.

Theorem 10.

Every uniform space uX has the Wallman realcompactification $\upsilon_u X$, contained in a $\beta-$like compactification $\beta_u X$ with the next equivalent properties:

(I) Every $coz-$mapping f from uX into any $\mathbb{R}-z_u-$complete space νR has a continuous $coz-$extension $\tilde{f}$ from $\upsilon_u X$ into νR.

(II) uX is C_u-embedded in $\upsilon_u X$.

(III) If a countable family of $u-$closed sets in uX has empty intersection, then their closures in $\upsilon_u X$ have empty intersection.

Continuation of Theorem 10.

(IV) The equality $\cap_{n\in\mathbb{N}}[Z_n]_{\upsilon_u X} = [\cap_{n\in\mathbb{N}} Z_n]_{\upsilon_u X}$ holds for any countable family of $u-$closed sets $\{Z_n\}_{n\in\mathbb{N}}$ in uX.

(V) Every point of $\upsilon_u X$ is the limit of a unique countably centered z_u-ultrafilter.

(VI) $\upsilon_u X$ is a completion of X with respect to a uniformity u^z_ω (u^z_ω has a base of all countable $u-$open coverings).

(VII) $\upsilon_u X$ is a completion of X with respect to a uniformity u^z_c (u^z_c is the smallest uniformity for which all functions from $C_u(X)$ are uniformly continuous).

Wallman realcompactification $\upsilon_u X$ is unique in the next sense: if a uniform space vY is a realcompactification of uX satisfies anyone of listed conditions, then there exists a $coz-$homeomorphism of $\upsilon_u X$ onto vY that leaves X pointwise fixed.

The next definition was given by [Z. Frolik, 1975].
Remind, that

Definition 11. [Z. Frolik, 1975]

A mapping $f : uX \to vY$ between uniform spaces is said to be a $coz-$*homeomorphism*, if f is a $coz-$mapping of uX onto vY in a one-to-one way, and the inverse mapping $f^{-1} : vY \to uX$ is a $coz-$mapping. A uniform spaces uX and vY are $coz-$*homeomorphic*, if there exists a $coz-$homeomorphism of uX onto vY.

Main Results

The next theorem demonstrates a relation between $\mathbb{R}-z_u-$completeness of uniform space uX and Wallman realcompactification $\upsilon_u X$.

Theorem 12.

For a uniform space uX the next conditions are equivalent:

(1) uX is $\mathbb{R}-z_u-$complete;

(2) X is complete with respect to a uniformity u^z_ω (u^z_ω has a base of all countable $u-$open coverings);

(3) X is complete with respect to a uniformity u^z_c (u^z_c is the smallest uniformity for which all functions from $C_u(X)$ are uniformly continuous).

(4) $uX = \upsilon_u X$;

(5) uX is $coz-$homeomorphic to a closed uniform subspace of a power of $u_{\mathbb{R}}\mathbb{R}$.

Main Results

We list the next properties of $\mathbb{R}-z_u-$complete uniform spaces.

Theorem 13.

(1) If X is a realcompact and non-Lindelöf space, then there exists a uniformity u on X such that uX is not $\mathbb{R}-z_u-$complete. The uniform space uX is C_u-embedded, but it is not $C-$embedded in $\upsilon_u X$.

(2) A Tychonoff space X is Lindelöf if and only if uX is $\mathbb{R}-z_u-$complete for any uniformity u on X.

(3) Every open uniform subspace of the $\aleph_0-$bounded metrizable uniform space is $\mathbb{R}-z_u-$complete.

(4) A closed uniform subspace of a $\mathbb{R}-z_u-$complete space is $\mathbb{R}-z_u-$complete.

(5) A product of any collection of $\mathbb{R}-z_u-$complete spaces is $\mathbb{R}-z_u-$complete if and only if every factor is $\mathbb{R}-z_u-$complete.

Continuation of Theorem 13

(6) A limit of an inverse system consisting of $\mathbb{R}-z_u-$complete spaces and "short" projections, being *coz*$-$mappings, is $\mathbb{R}-z_u-$complete.

(7) Let $\{u_t X_t : t \in T\}$ be a collection of $\mathbb{R}-z_u-$complete uniform subspaces of $\mathbb{R}-z_u-$complete space uX, i.e. $u_t = u|_{X_t}$ for any $t \in T$. Then the intersection $\cap\{X_t : t \in T\} = Y$, equipped by the uniformity $v = u|_Y$, is $\mathbb{R}-z_u-$complete.

(8) If $f : uX \to vY$ is *coz*$-$perfect mapping of uX onto $\mathbb{R}-z_u-$complete uniform space vY, then uX is $\mathbb{R}-z_u-$complete.

(9) An $u-$open subspace of $\mathbb{R}-z_u-$complete space uX is $\mathbb{R}-z_u-$complete.

(10) $\mathbb{R}-z_u-$complete and C_u-embedded subspace is closed.

Main Results

Under *coz−perfect* mapping we will understand the next:

Definition 14.

A $coz-$mapping $f : uX \to vY$ between uniform spaces is said to be $coz-$*perfect*, if 1) f is closed, and 2) f is compact, i.e. $f^{-1}(y)$ is a compactum in X for any point $y \in Y$.

We note, that every $coz-$mapping $f : uX \to vY$ has $\beta-$like extension $\beta_u f : \beta_u X \to \beta_v Y$ [Chekeev, 2016].

In the category *ZUnif* $coz-$perfect mappings have the next inner and categorical characterizations.

Theorem 15.

Let uX and vY be a uniform spaces. Then for $coz-$mapping $f : uX \to vY$ the next conditions are equivalent:

(1) f is $coz-$perfect.

(2) If p is z_u-ultrafilter on uX and prefilter $f(p) = \{f(Z) : Z \in p\}$ is converging to point $y \in Y$, then p is converging to point $x \in f^{-1}(y)$.

(3) For extension mapping $\beta_u f : \beta_u X \to \beta_v Y$ a remainder $\beta_u X \setminus X$ transfers to a remainder $\beta_v Y \setminus Y$, i.e. $\beta_u f(\beta_u X \setminus X) \subset \beta_v Y \setminus Y$.

(4) Square

$$\begin{array}{ccc} uX & \xrightarrow{i_X} & \beta_u X \\ f \downarrow & & \downarrow \beta_u f \\ vY & \xrightarrow[i_Y]{} & \beta_v Y \end{array}$$

is pullback in category *ZUnif*, where i_X and i_Y are $coz-$homeomorphic embeddings.

Main Results

By analogy with work [Curzer-Hager, 1976] we introduce a concept of *locally finitely additive* z_u*−ultrafilter*.

Definition 16.

Let p be a z_u-ultrafilter and $co(p) = \{X \setminus Z : Z \in p\}$ be an $u-$open family. A family $co(p)$ is said to be *locally finitely additive*, if $\cup\alpha \in co(p)$ whenever $\alpha \subset co(p)$ and α is *locally finite*. Every z_u-ultrafilter p such that $co(p)$ is locally finitely additive, is said to be *weakly Cauchy* z_u*−ultrafilter*.

The name of weakly Cauchy z_u-ultrafilter in the Definition 16 is due to that every Cauchy z_u-ultrafilter with respect uniformity u^z_{cf} satisfies to the locally finitely additive property and it is countably centered and vise versa.

Definition 17.

A uniform space uX is said to be *weakly* z_u*−complete*, if every weakly Cauchy z_u-ultrafilter converges.

Main Results

Wallman completeness of uniform spaces will be correspond to the weakly z_u-completeness.

Definition 18.

Wallman completion $\mu_u X$ of a uniform space uX is the subspace of $\beta_u X$ consisting of the set of all weakly Cauchy z_u-ultrafilteres on $\mathcal{Z}_u$.

The next theorem characterizes Wallman completion $\mu_u X$ of uniform space uX.

Theorem 19.

Every uniform space uX has Wallman completion $\mu_u X$, contained in a $\beta-$like compactification $\beta_u X$ with the next equivalent properties:

(I) Every *coz*$-$mapping f from uX into any weakly z_u-complete uniform space vY has a *coz*$-$mapping extension $\tilde{f}$ from $\mu_u X$ into vY.

(II) Every *coz*$-$mapping f from uX into an arbitrary metric uniform space $u_\rho M$ has a *coz*$-$mapping extension $\tilde{f}$ from $\mu_u X$ into $u_\rho M$.

(III) If $\{Z_i\}_{i\in I}$ is a family of $u-$closed sets with $\{X \setminus Z_i\}_{i\in I}$ locally finite, and $\cap_{i\in I} Z_i = \emptyset$, then $\cap_{i\in I}[Z_i]_{\mu_u X} = \emptyset$.

Continuation of Theorem 19.

(IV) If $\{Z_i\}_{i\in I}$ is a family of $u-$closed sets with $\{X \setminus Z_i\}_{i\in I}$ locally finite, then $\cap_{i\in I}[Z_i]_{\mu_u X} = [\cap_{i\in I} Z_i]_{\mu_u X}$.

(V) Every point of $\mu_u X$ is the limit of unique weakly Cauchy z_u-ultrafilter.

(VI) $\mu_u X$ is a completion of X with respect to a uniformity u^z_{cf} (u^z_{cf} has a base of all locally finite $coz-$additive $u-$open coverings).

Wallman completion $\mu_u X$ is unique in the next sense: If a uniform space vY is an extension of uX satisfies anyone of listed conditions, then there exists a $coz-$homeomorphism of $\mu_u X$ onto vY that leaves X pointwise fixed.

Main Results

Below we have the next characterizations of weakly z_u-complete uniform spaces.

Theorem 20.

For a uniform space uX the next conditions are equivalent:

(1) uX is weakly z_u-complete;

(2) X is complete with respect to a uniformity u^z_{cf} (u^z_{cf} has a base of all locally finite $coz-$additive $u-$open coverings);

(3) $uX = \mu_u X$;

(4) uX is $coz-$homeomorphic to a closed uniform subspace of metric uniform spaces product.

The next properties of weakly z_u-complete uniform spaces hold.

Theorem 21.

(1) Every metric uniform space is a weakly z_u-complete.

(2) A closed uniform subspace of a weakly z_u-complete space is a weakly z_u-complete.

(3) A product of any collection of a weakly z_u-complete spaces is a weakly z_u-complete iff every factor is a weakly z_u-complete.

(4) A limit of an inverse system consisting of a weakly z_u-complete spaces and "short" projections, being *coz*$-$mappings, is a weakly z_u-complete.

(5) Let $\{u_t X_t : t \in T\}$ be a collection of a weakly z_u-complete uniform subspaces of a weakly z_u-complete space uX, i.e. $u_t = u|_{X_t}$ for any $t \in T$. Then the intersection $\cap\{X_t : t \in T\} = Y$, equipped by the uniformity $v = u|_Y$, is a weakly z_u-complete.

(6) If $f : uX \to vY$ is *coz*$-$perfect mapping of uX onto a weakly z_u-complete uniform space vY, then uX is a weakly z_u-complete.

Conclusions

All of the foregoing leads us to the following conclusions in the category *ZUnif*.
Every compactum (or z_u-complete uniform space) is a closed subspace of power of $I = [0, 1]$, hence a class $\mathcal{K}$ of compacta in the category *ZUnif* coincides with epi-reflective hull $\mathfrak{L}([0, 1])$ [Franklin, 1971; Herrlich, 1971; Hager, 1975]. For any uniform space uX $\beta-$like compactification $\beta_u X$ is a projective object in $\mathfrak{L}([0, 1])$, i.e. $\beta_u X$ is the essentially unique compactum containing X densely such that each $coz-$mapping $f : uX \to K$ $(K \in \mathfrak{L}([0, 1]))$ admits a continuous extension $\beta_u f : \beta_u X \to K$, or $\beta_u : uX \to \beta_u X$ is an epi-reflection and $coz-$homeomorphic embedding.

Every realcompact space (or $\mathbb{R}-z_u-$complete uniform space) is a closed subspace of power of $u_{\mathbb{R}}\mathbb{R}$, hence a class $\mathcal{R}$ of realcompact spaces in the category *ZUnif* coincides with epi-reflective hull $\mathfrak{L}(u_{\mathbb{R}}\mathbb{R})$ [Franklin, 1971; Herrlich, 1971; Hager, 1975]. For any uniform space uX Wallman realcompactification $\upsilon_u X$ is a projective object in $\mathfrak{L}(u_{\mathbb{R}}\mathbb{R})$, i.e. $\upsilon_u X$ is the essentially unique realcompact space (or $\mathbb{R}-z_u-$complete uniform space) containing X densely such that each $coz-$mapping $f : uX \to vY$ ($vY \in \mathfrak{L}(u_{\mathbb{R}}\mathbb{R})$) admits a $coz-$mapping extension $\upsilon_u f : \upsilon_u X \to vY$, or $\upsilon_u : uX \to \upsilon_u X$ is an epi-reflection and $coz-$homeomorphic embedding.

Let $\mathcal{M}$ be a class of all metric uniform spaces. Every weakly z_u-complete uniform space is a closed uniform subspace of product from the class $\mathcal{M}$, hence a class of all weakly z_u-complete uniform spaces in the category *ZUnif* coincides with epi-reflective hull $\mathfrak{L}(\mathcal{M})$ [Franklin, 1971; Herrlich, 1971; Hager, 1975]. For any uniform space uX Wallman completion $\mu_u X$ is a projective object in $\mathfrak{L}(\mathcal{M})$, i.e. $\mu_u X$ is the essentially unique weakly z_u-complete uniform space containing X densely such that each $coz-$mapping $f : uX \to vY$ $(vY \in \mathfrak{L}(\mathcal{M}))$ admits a $coz-$mapping extension $\mu_u f : \mu_u X \to vY$, or $\mu_u : uX \to \mu_u X$ is an epi-reflection and $coz-$homeomorphic embedding.

From Theorems 10, 12, 13 it follows that for every $f \in C_u(X)$ we have an extension mapping $\beta_u f : \beta_u X \to s_{u_{\mathbb{R}}}\mathbb{R}$, where $s_{u_{\mathbb{R}}}\mathbb{R}$ is Samuel compactification of $u_{\mathbb{R}}\mathbb{R}$, and

$$\upsilon_u X = \cap\{(\beta_u f)^{-1}(\mathbb{R}) : f \in C_u(X)\}.$$

From Theorems 19, 20, 21 it follows that for every $coz-$mapping $f : uX \to u_\rho M$, $u_\rho M \in \mathcal{M}$, we have an extension mapping $\beta_u f : \beta_u X \to s_{u_\rho} M$, where $s_{u_\rho} M$ is Samuel compactification of $u_\rho M$ and

$$\mu_u X = \cap\{(\beta_u f)^{-1}(M) : f : uX \to u_\rho M \text{ is a } coz-\text{mapping}, \ u_\rho M \in \mathcal{M}\}.$$

Hence for any uniform space uX it holds

$$X \subset \mu_u X \subset \upsilon_u X \subset \beta_u X.$$

If $u = u_f$ is a fine uniformity, then for Tychonoff spaces the well-known chain [Morita, 1970; Curzer–Hager, 1976] of inclusions holds:

$$X \subset \mu X \subset \upsilon X \subset \beta X,$$

where μX is Dieudonne completion, υX is Hewitt extension, βX is Stone–Čech compactification.
We note, that epi-reflective hull $u_{\mathbb{R}}\mathbb{R}$ in category *Unif* coincides with class of realcomplete spaces in sense [Hušek–Pulgarin, 2015].

In case $u = u_f$ is a fine uniformity, we have the next correspondences table of categories *ZUnif* and *Tych*.

ZUnif	**Tych**
z_u-completeness	$z-$completeness$\Leftrightarrow$compactness
$\mathbb{R}-z_u-$completeness	$\mathbb{R}-z-$completeness$\Leftrightarrow$realcompactness
weakly z_u-completeness	weakly $z-$completeness $\Updownarrow$ Dieudonne completeness

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