The hyperspace of large order arcs

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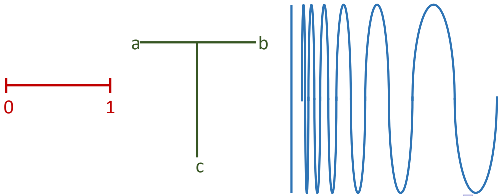
Definitions

Definition

A continuum is a compact connected metric space.

Examples

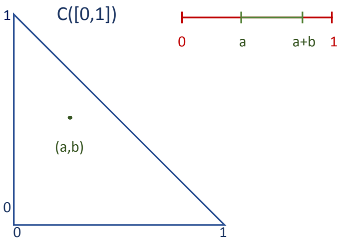
The unit interval $[0, 1]$, a simple triod, the closure of the graph $\sin(\frac{1}{x})$.



Hyperspace of subcontinua

Definition

Given a continuum X, let $C(X)$ be the hyperspace of subcontinua of X, consisting of all subcontinua of X. We let $C(X)$ be metrized with the Hausdorff metric.



Hausdorff metric

Definition

Let X be a continuum with metric d, given $A \in C(X)$ and $\varepsilon > 0$, the neighbourhood of radius ε centered in A is defined as the set $N_\varepsilon(A) = \bigcup\{B_\varepsilon(a) : a \in A\}$, where $B_\varepsilon(a)$ is the open ball in X of radius ε centered in a.

If a continuum X consists of only one point, we say that X is degenerate, and if X consists of more than one point, we say that X is non-degenerate.

Definition

Given a continuum X and $A, B \in C(X)$, the Hausdorff metric H in $C(X)$ is defined for each $A, B \in C(X)$ by
$H(A, B) = \inf\{\varepsilon > 0 : A \subset N_\varepsilon(B) \text{ and } B \subset N_\varepsilon(A)\}$.

Whitney maps

Definition

Let X be a continuum with more than one point. A map $\mu : C(X) \to [0,1]$ is a Whitney map if the following conditions hold:

- $\mu(X) = 1$ and $\mu(\{x\}) = 0$ for each $x \in X$,
- if $A, B \in C(X)$ and $A \subsetneq B$, then $\mu(A) < \mu(B)$.

Theorem

Let X be a continuum with more than one point. Then there exists a Whitney map $\mu : C(X) \to [0,1]$.

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Theorem

Let X be a continuum with more than one point. Then there exists a Whitney map $\mu : C(X) \to [0,1]$.

Order arcs

Definition

An order arc in $C(X)$ is a subcontinuum $\mathcal{O} \subset C(X)$ homeomorphic to an arc, such that for each $A, B \in \mathcal{O}$, we have that $A \subset B$ or $B \subset A$.

We also call the degenerate subcontinua of $C(X)$ order arcs.

Theorem

Let X be a continuum and $A, B \in C(X)$ such that $A \subset B$. Then there exists an order arc $\mathcal{O} \subset C(X)$ that joins A to B.

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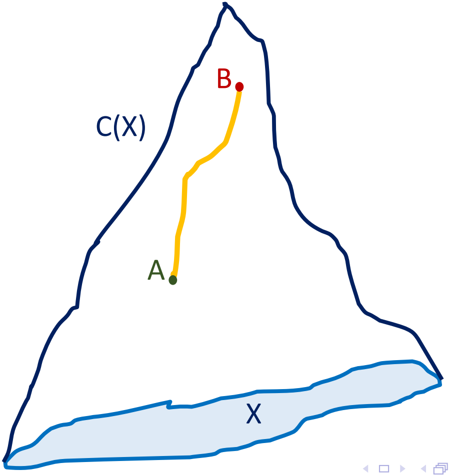
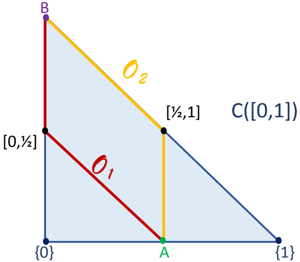
Order arc joining A to B



Examples of orders arcs

Let $X = [0,1]$, $A = \{\frac{1}{2}\}$ and $B = [0,1]$. Define the sets $\mathcal{O}_1 = \{[t, \frac{1}{2}] : 0 \leq t \leq \frac{1}{2}\} \cup \{[0,t] : \frac{1}{2} \leq t \leq 1\}$ and $\mathcal{O}_2 = \{[\frac{1}{2}, t] : \frac{1}{2} \leq t \leq 1\} \cup \{[t,1] : 0 \leq t \leq \frac{1}{2}\}$, then $\mathcal{O}_1$ and $\mathcal{O}_2$ are two distinct order arcs joining A to B.



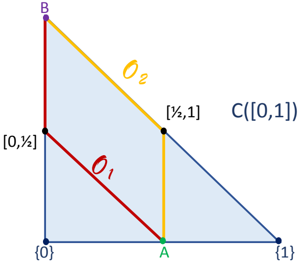
Order arcs

The set of all order arcs $OA(X)$ of a continuum X was studied by Curtis and Lynch for locally connected continua. They characterized those continua X such that $OA(X)$ is homeomorphic to a Hilbert cube. The showed that if X is the union of a circle and an interval at the middle point of the interval, then $OA(X)$ is a Hilbert cube. We see that taking the space $OA(X)$ loses information about the space X.

Large order arcs

Definition

Given a continuum X, a large order arc in $C(X)$ is an order arc in $C(X)$ that joins X to an element of the form $\{x\}$, for some $x \in X$.



Basic properties of large order arcs

Proposition

Let X be a continuum, $x \in X$ and $\mathcal{A}$ an order that in $C(X)$ that contains $\{x\}$ and X. Then the following properties hold:

- if $\{y\} \in \mathcal{A}$ for some $y \in X$, then $x = y$,
- given a Whitney map $\mu : C(X) \to [0, 1]$, then $\mu(\mathcal{A}) = [0, 1]$ and μ is a homeomprhism between $\mathcal{A}$ and $[0, 1]$,
- the endpoints of $\mathcal{A}$ are X and $\{x\}$,
- if $\mathcal{B}$ is an order arc in $C(X)$ such that $\mathcal{A} \subset \mathcal{B}$ then $\mathcal{A} = \mathcal{B}$.

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Definitions

Definition

Given a continuum X and $x \in X$, let $LOA(X)$ be the hyperspace of all large order arcs in $C(X)$, and let $LOA(x, X)$ be the hyperspace of all large order arcs that contain the element $\{x\}$.

We consider $LOA(X)$ and $LOA(x, X)$ as subspaces of $C(C(X))$.

Proposition

Let X be a continuum and $x \in X$. Then $LOA(x, X)$ and $LOA(X)$ are non-empty continua.

Proposition

$LOA(X) = \bigcup_{x \in X} LOA(x, X)$.

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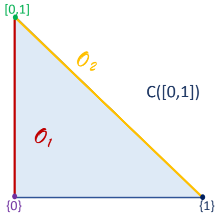
$LOA(x, X)$ can be degenerate

Let $X = [0, 1]$ and $x = 0$ or 1, then $LOA(x, [0, 1])$ is degenerate. More specifically, $\mathcal{O}_1 = \{[0, t] : 0 \leq t \leq 1\}$ is the only element of $LOA(\{0\}, [0, 1])$, and $\mathcal{O}_2 = \{[t, 1] : 0 \leq t \leq 1\}$ is the only element of $LOA(\{1\}, [0, 1])$.



Theorem[Chacón-Tirado]

Let X be a continuum and $x \in X$. Then $LOA(x, X)$ and $LOA(X)$ are closed subspaces of $C(C(X))$.

Theorem[Chacón-Tirado]

Let X be a continuum and $x \in X$. Then $LOA(x, X)$ is an arcwise connected continuum, and $LOA(X)$ is a continuum.

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Absolute retract

Definition

Let $X \subset Y$ topological spaces. We say that X is a retract of Y if there exists a retractions $r : Y \to X$, that is, r is a map such that $r(x) = x$ for each $x \in X$.

Definition

We say that a topological space X is an absolute retract(AR) if whenever X is embedded as a closed subspace of a space Y, then X is a retract of Y.

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$LOA(x, X)$ is an AR

Theorem[Chacón-Tirado]

Let X be a continuum and $x \in X$. Then $LOA(x, X)$ is an AR.

Definition

A X is called:

- decomposable if X can be represented as the union of two proper subcontinua of X.
- indecomposable if X is not decomposable, and
- hereditarily indecomposable if each subcontinuum of X is indecomposable.

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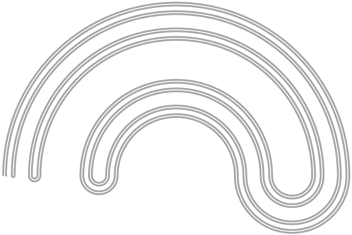
Examples

Knaster buckethandle is a indecomposable continuum:



The pseudo-arc is a hereditarily indecomposable continuum.

When $LOA(x, X)$ is degenerate

Theorem[Chacón-Tirado]

Let X be a continuum and $x \in X$. Then $LOA(x, X)$ is degenerate if and only if for each $A, B \in C(X)$ such that $x \in A \cap B$, we have that $A \subset B$ or $B \subset A$.

Corollary

Let X be a continuum. Then $LOA(x, X)$ is degenerate for each $x \in X$ if and only if X is hereditarily indecomposable.

Corollary

If X is a hereditarily indecomposable continuum, then $LOA(X)$ is homeomorphic to X.

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If X is a hereditarily indecomposable continuum, then $LOA(X)$ is homeomorphic to X.

Definition

A closed subset Y in a compact metric space X is called a *Z-set* if for each $\varepsilon > 0$ there exists a map $f : X \to X \setminus Y$ such that $d(x, f(x)) < \varepsilon$ for each $x \in X$.

Definition

A map $f : X \to X$ is called *Z-map* if its image is a Z-set.

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When $LOA(x, X)$ is non-degenerate

Theorem[Toruńczyk]

Let X be an AR. If the identity map on X is uniform limit of Z-maps, then X is homeomorphic to the Hilbert cube.

Theorem[Chacón-Tirado]

Let X is a continuum and $x \in X$. If $LOA(x, X)$ is non-degenerate, then the identity map on $LOA(x, X)$ is uniform limit of Z-maps, then by Toruńczyk, $LOA(x, X)$ is homeomorphic to the Hilbert cube.

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More on $LOA(x, X)$

We consider the metric on $LOA(x, X)$ as the induced by the Hausdorff metric on $C(C(X))$.

Theorem[Chacón-Tirado]

Let $LOA(x, X)$ be metrized with the Hausdorff metric on $C(C(X))$. Then the open balls in $LOA(x, X)$ are arcwise connected.

When X is an AR

Theorem[Chacón-Tirado]

If X is an AR, then $LOA(X)$ is an AR.

Theorem[Chacón-Tirado]

if X is an AR, then the identity map on $LOA(X)$ is a uniform limit of Z-maps.

Corollary

If X is an AR, then $LOA(X)$ is homeomorphic to the Hilbert cube.

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Topological groups

Definition

A topological group is a topological space X endowed with a group operation $\cdot : X \times X \to X$ such that $\cdot$ and the inverse are continuous.

Definition

A continuum X is called homogeneous if for each $x, y \in X$ there exists a homeomorphism $h : X \to X$ such that $h(x) = y$.

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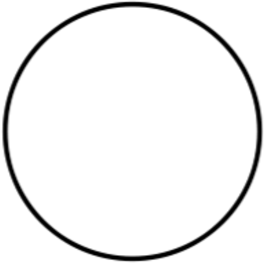
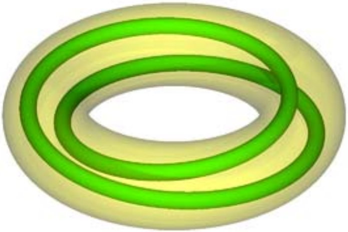
Examples

The unit circle, products of circles, dyadic solenoids...





Topological groups

Theorem[Chacón-Tirado]

Let X be a topological group and $x \in X$. Then $LOA(X)$ is homeomorphic to $X \times LOA(x, X)$.

Corollary[Chacón-Tirado]

Let S^1 be the unit circle. Then $LOA(S^1)$ is homeomorphic to $S^1 \times Q$, where Q is the Hilbert cube.

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Theorem

If X is a topological group, then $LOA(X)$ is homogeneous.

Question

If X is homogeneous, is it true that $LOA(X)$ is homogeneous?.

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Relation between properties of X and properties of $LOA(X)$

Theorem[Chacón-Tirado]

$LOA(X)$ is arcwise connected if and only if X is arcwise connected.

Theorem[Chacón-Tirado]

$LOA(X)$ is locally connected if and only if X is locally connected.

Relation between properties of X and properties of $LOA(X)$

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Theorem[Chacón-Tirado]

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Theorem[Chacón-Tirado]

The fundamental groups of X and of $LOA(X)$ are isomorphic.

Theorem[Chacón-Tirado]

Let X be a contractible continuum. Then $LOA(X)$ is contractible.

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Connectedness im kleinen

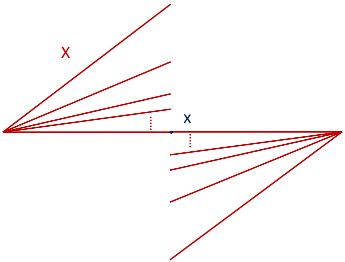
Definition

A continuum X is called connected im kleinen (cik) at a point $x \in X$ if for each $\varepsilon > 0$ there exists a subcontinuum of X with diameter less than ε that contains x in its interior.

Theorem[Chacón-Tirado]

Let X be a continuum cik at $x \in X$. Then for each $\mathcal{L} \in LOA(x, X)$ we have that $LOA(X)$ is cik at $\mathcal{L}$.

The converse is not true. Consider X and x as in the picture below, then X is not cik at x, and $LOA(X)$ is cik at any point $\mathcal{L} \in LOA(x, X)$.



Aposyndesis

Aposyndesis is a separation property weaker than connectedness im kleinen.

Definition

A continuum X is called aposyndetic if for each $p, q \in X$, with $p \neq q$, there exists a subcontinuum of X that contains p in its interior, and does not contain q.

Theorem[Chacón-Tirado]

Let X be aposyndetic. Then $LOA(X)$ is aposyndetic.

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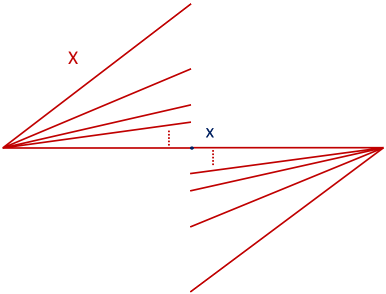
Theorem[Chacón-Tirado]

Let X be aposyndetic. Then $LOA(X)$ is aposyndetic.

Conjecture

We believe that the same example as before shows that the converse of the previous theorem is not true, $LOA(X)$ is aposyndetic while X is not.



Fixed point property

Definition

A continuum X has the fixed point property (FPP) if each map $f : X \to X$ has a fixed point.

Theorem[Chacón-Tirado]

If X is a continuum such that $LOA(X)$ has the FPP, then X has the FPP.

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Fixed point property

Since absolute retracts have the FPP, we have the following theorem

Theorem

Let X be an absolute retract. Then $LOA(X)$ has the FPP.

Fixed point property

Theorem[Chacón, Herrera, Macías]

Let X be a chainable continuum such that each arc-component is compact. Then $LOA(X)$ has the FPP.

Question

Let X be a continuum with the FPP. Is it true that $LOA(X)$ has the FPP?

Induced maps

In the present section, let X, Y be continua and $f : X \to Y$ is a surjective mapping. Let us remember that the induced map $C(f) : C(X) \to C(Y)$ is defined by $C(f)(A) = f(A)$, for each $A \in C(X)$.

Definition

The induced map $LOA(f) : LOA(X) \to LOA(Y)$ is defined for each $\mathcal{L} \in LOA(X)$ by $LOA(f)(\mathcal{L}) = \{f(L) : L \in \mathcal{L}\}$.

Since f is surjective, then $LOA(f)$ is well defined, and since $LOA(f)$ is just the restriction of the induced map $C(C(f))$, then $LOA(f)$ is continuous.

Definitions

Definition

The map $f : X \to Y$ is called weakly confluent is its induced map $C(f)$ is surjective, and f is called confluent if for each $B \in C(Y)$ and each component A of $f^{-1}(B)$, we have that $f(A) = B$.

Theorem

If the map $LOA(f)$ is surjective, then f is weakly confluent.

Theorem

If f is confluent, then $MOA(f)$ is surjective.

Definitions

Definitions

The map f is called monotone(light) if $f^{-1}(y)$ is connected (totally disconnected) for each $y \in Y$.

Theorem

If f is monotone, then $LOA(f)$ is monotone.

Theorem

If $LOA(f)$ is injective, then f is light.

Theorem

Let $f : [0,1] \to [0,1]$ be an onto map such that $LOA(f)$ is light. Then f is a homeomorphism.

Theorem

Let X be a continuum $f : X \to [0,1]$ be an onto map such that $LOA(f)$ is light. Then f is a homeomorphism.

THANK YOU