On quasi–convex null sequences

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Notation

A subset C of a real vector space V is called **convex** if

$$\lambda x + (1-\lambda)y \in C \quad \forall \lambda \in [0,1],\ \forall x, y \in C.$$

In particular, if $x, y \in C$ are two different points, then $\{\lambda x + (1-\lambda)y : \lambda \in [0,1]\} \subseteq C$ and hence $|C| \geq \mathfrak{c}$.

In order to define "**convex**" sets in an abelian topological group, we use the description of closed, symmetric convex subsets of real locally convex vector spaces given by the Hahn Banach theorem:

Theorem

Let V be a real locally convex vector space and $0 \in C \subseteq V$. Then the following assertion are equivalent:

1. *C is closed, symmetric and convex.*
2. *For every $x \notin C$ there exists a continuous linear form $f : V \to \mathbb{R}$ such that*

$$f(C) \subseteq \left[-\frac{1}{4}, \frac{1}{4}\right] \quad \text{and} \quad |f(x)| > \frac{1}{4}.$$

Notation

The torus $\mathbb{T} := \mathbb{R}/\mathbb{Z}$, $\quad \mathbb{T}_+ = [-\frac{1}{4}, \frac{1}{4}] + \mathbb{Z}$

Definition

Let (G, τ) be an abelian topological group.

$$G^\wedge := (G, \tau)^\wedge := \{\chi : G \to \mathbb{T} | \ \chi \text{ is a continuous hom.}\}$$

is under pointwise addition an abelian group. It is called **dual group** or **character group**.

Quasi–convex sets

Definition

For a subset A of a topological group (G,ρ) we define the **polar** of A as

$$A^{\triangleright} := \{\chi \in G^{\wedge} \mid \forall\, x \in A \;\; \chi(x) \in \mathbb{T}_+\}$$

and for $B \subseteq G^{\wedge}$ we define the **pre–polar** of B by

$$B^{\triangleleft} := \{x \in G \mid \forall\, \chi \in B \;\; \chi(x) \in \mathbb{T}_+\}.$$

A subset A of a topological group (G,τ) is called **quasi–convex** if for every $x \in G \setminus A$ there exists a character $\chi \in A^{\triangleright}$ such that $\chi(x) \notin \mathbb{T}_+$.

Locally quasi–convex groups

Definition (Vilenkin; 1951)

A topological group (G, τ) is called **locally quasi–convex** if it has a neighborhood basis at 0 consisting of quasi–convex sets.

Examples of quasi–convex sets

Example

1. $\mathbb{T}_+ \subseteq \mathbb{T}$ is quasi–convex.
2. $\mathbb{T}_m := \left[-\frac{1}{4m}, \frac{1}{4m}\right] + \mathbb{Z} \subseteq \mathbb{T}$ is quasi–convex.
3. The intersection of quasi–convex sets is quasi–convex.
4. The inverse image of a quasi–convex set under a continuous homomorphism is quasi–convex.
5. For $B \subseteq G^\wedge$ the set
$$(B, \mathbb{T}_m) := \bigcap_{\chi \in B} \chi^{-1}(\mathbb{T}_m)$$
is quasi–convex.
6. For every $A \subseteq G$ the set $(A^\triangleright)^\triangleleft = (A^\triangleright, \mathbb{T}_+)$ is quasi–convex.

The quasi–convex hull

Proposition

For a subset A of an abelian topological group (G, τ) the set

$$(A^{\triangleright})^{\triangleleft}$$

is the smallest quasi–convex set containing A. It is called the **quasi–convex hull** *of A and denoted by $\mathrm{qc}(A)$.*

Corollary

$A \subseteq G$ is quasi–convex iff $A = (A^{\triangleright})^{\triangleleft}$.

Examples of locally quasi–convex groups

Example

1. A Hausdorff topological vector space is locally convex iff it is locally quasi–convex.
2. Every character group endowed with the compact–open topology is locally quasi–convex.
3. Every locally compact abelian (LCA for short) group is locally quasi–convex.

Cardinality of quasi–convex sets

Proposition

Every symmetric, closed and convex subset of a locally convex vector space is locally quasi–convex. Hence there exist quasi–convex sets of cardinality $\geq \mathfrak{c}$.

Proposition (L.A. 1998)

If G is an MAP group, then

$$\mathrm{qc}(x) = \{x, -x, 0\}$$

for every $x \in G$.

Proposition (L.A. 1998; Dikranjan, Kunen 2007)

If G is an MAP group, then the quasi–convex hull of every finite subset is finite.

Question

Question

Are there countably infinite quasi–convex sets?

Quasi–convex null sequences

Definition

A sequence $(x_n)_{n\in\mathbb{N}}$ in an abelian topological group is called a **quasi–convex null sequence** if

$$x_n \to 0$$

and the set

$$\{x_n : \ n\in\mathbb{N}\}\cup\{-x_n : \ n\in\mathbb{N}\}\cup\{0\}$$

is quasi–convex.

Question

1. *Are there quasi–convex null sequences?*
2. *Which (LCA) groups have quasi–convex null sequences?*

Quasi–convex null sequences in $\mathbb{T}$

Theorem (L. de Leo 2008)

Let $(a_n) \in \mathbb{N}^{\mathbb{N}}$ with $a_{n+1} - a_n \geq 2$ for all $n \in \mathbb{N}$. Then

$$(2^{-a_n+1} + \mathbb{Z})_{n\in\mathbb{N}}$$

is a quasi-convex null sequence in $\mathbb{T}$.

Quasi–convex null sequences in $\mathbb{R}$ and $\mathbb{J}_2$

Theorem (D. Dikranjan, L. de Leo, 2010)

Let $(a_n) \in \mathbb{N}^{\mathbb{N}}$ with $a_{n+1} - a_n \geq 2$ for all $n \in \mathbb{N}$. Then

$$(2^{-a_n+1})_{n\in\mathbb{N}}$$

is a quasi-convex null sequence in $\mathbb{R}$.

Example

$$\mathrm{qc}(\{2^{-n} : \; n \in \mathbb{N}_0\}) = [-1, 1] \subseteq \mathbb{R}$$

Theorem (D. Dikranjan, L. de Leo, 2010)

Let $(a_n) \in \mathbb{N}^{\mathbb{N}}$ with $a_{n+1} - a_n \geq 2$ for all $n \in \mathbb{N}$. Then

$$(2^{a_n-1})_{n\in\mathbb{N}}$$

is a quasi-convex null sequence in $\mathbb{J}_2$.

Quasi–convex sets in bounded groups

Proposition (D. Dikranjan, G. Lukács; 2010)

Every abelian topological group of exponent ≤ 3 has no non–trivial quasi–convex null sequence.

Proof.

Let G be a bounded abelian topological group of exponent ≤ 3. Fix $x \in G$ and $\chi \in G^\wedge$.If $\chi(x) \neq 0 + \mathbb{Z}$, then $\chi(x) \notin \mathbb{T}_+$. This implies $\{x\}^\triangleright = \{x\}^\perp$. Hence, if (x_n) is a null sequence, then $\{x_n : \ n \in \mathbb{N}\}^\triangleright$ is a subgroup of $G^\wedge$ and so is $\mathrm{qc}(\{x_n : \ n \in \mathbb{N}\}) = (\{x_n : \ n \in \mathbb{N}\}^\perp)^\triangleleft$. In particular, $\mathrm{qc}(\{x_n : \ n \in \mathbb{N}\})$ is a subgroup of G, hence a homogeneous space. This yields
$\{0\} \cup \{\pm x_n : \ n \in \mathbb{N}\} \neq \mathrm{qc}(\{x_n : \ n \in \mathbb{N}\})$. □

The groups $\mathbb{Z}_2^\kappa$ and $\mathbb{Z}_3^\kappa$

Corollary (D. Dikranjan, G. Lukács; 2010)

The groups $\mathbb{Z}_2^\kappa$ and $\mathbb{Z}_3^\kappa$ do not admit a non–trivial quasi–convex null sequence.

Quasi–convex null sequences in LCA groups

Theorem (D. Dikranjan, G. Lukács; 2010)

For a LCA group G the following assertions are equivalent:

1. *G has no non–trivial quasi–convex null sequence.*
2. *Either the subgroup* $G[2] = \{x \in G : \ 2x = 0\}$ *or* $G[3] = \{x \in G : \ 3x = 0\}$ *is open in G.*
3. *G contains a compact open subgroup topologically isomorphic to* $\mathbb{Z}_2^\kappa$ *or* $\mathbb{Z}_3^\kappa$ *(*κ *a cardinal).*

Quasi–convex null sequences in LCA groups

Question

Do similar results hold for arbitrary precompact abelian groups?

Quasi–convex null sequences in precompact groups

Theorem (D. Dikranjan, G. Lukács; 2014)

If G is a **bounded** *precompact group or a* **minimal** *group then the following assertions are equivalent:*

1. *G has no non–trivial quasi–convex null sequences.*
2. *$G[2]$ or $G[3]$ is sequentially open.*

Theorem (D. Dikranjan, G. Lukács; 2014)

If G is an abelian **ω–bounded** *or a* **totally minimal** *group then the following assertions are equivalent:*

1. *G has no non–trivial quasi–convex null sequences.*
2. *$G[2]$ or $G[3]$ is open.*

The following question is still open:

Question (D. Dikranjan, G. Lukács; 2010)

Let H be an infinite cyclic subgroup of $\mathbb{T}$. Does H admit a non–trivial quasi–convex null sequence?

We will give a partial answer.

Notation

Let $(b_n)_{n\in\mathbb{N}_0}$ be a strictly increasing sequence of natural numbers such that $b_0 = 1$ and $b_n | b_{n+1}$ for all $n \in \mathbb{N}$; this means $q_n := \frac{b_n}{b_{n-1}} \in \mathbb{N}$ for all $n \in \mathbb{N}$.
We assume further that for all $n \in \mathbb{N}$

$$16 \cdot q_2 \cdot \ldots \cdot q_n | q_{n+1} \iff 16\frac{b_n}{b_1} | \frac{b_{n+1}}{b_n}.$$

Define

$$\alpha := \sum_{k=1}^{\infty} \frac{1}{b_k} + \mathbb{Z}$$

Theorem (L.A. 2016)

The group $\langle\alpha\rangle$ contains a quasi–convex null sequence; more precisely,

$$(b_n\alpha)_{n\in\mathbb{N}}$$

is a quasi–convex null sequence in $\langle\alpha\rangle$.
The set

$$S := \{0+\mathbb{Z}\}\cup\{\pm b_n\alpha :\ n\in\mathbb{N}\}$$

is even quasi–convex in $\mathbb{T}$.

Proof.

$$b_n\alpha = b_n \sum_{k=1}^{\infty} \frac{1}{b_k} + \mathbb{Z} = \underbrace{\sum_{k=n+1}^{\infty} \frac{b_n}{b_k}}_{=:x_n} + \mathbb{Z}$$

$$\mathrm{qc}(S) \subseteq \{\pm \sum_{k=2}^{\infty} \sum_{j=1}^{k-1} \varepsilon_{j,k} \frac{b_j}{b_k} : \ \varepsilon_{j,k} \in \{0,1\}\}$$

$$\mathrm{qc}(S) \subseteq \{\pm \sum_{k=2}^{\infty} \sum_{j=1}^{k-1} \varepsilon_j \frac{b_j}{b_k} : \ \varepsilon_j \in \{0,1\}\} =$$

$$= \{\sum_{j=1}^{\infty} \varepsilon_j \sum_{k=j+1}^{\infty} \frac{b_j}{b_k} : \ \varepsilon_j \in \{0,1\}\} =$$

$$= \{\pm \sum_{j=1}^{\infty} \varepsilon_j x_j : \ \varepsilon_j \in \{0,1\}\}$$

S is quasi–convex (in $\mathbb{T}$).

□

Theorem

There are $\mathfrak{c}$ many non–torsion elements α in $\mathbb{T}$ such that $\langle\alpha\rangle$ contains a non–trivial null sequence.

Open Questions

1. (de Leo) Does every null sequence in $\mathbb{T}$ ($\mathbb{R}$, in a LCA group G) contain a quasi–convex null sequence?
2. (de Leo; Dikranjan) For which sequences $(c_n) \in \mathbb{N}^{\mathbb{N}}$ is $\left(\frac{c_n}{2^{a_n}} + \mathbb{Z}\right)_{n \in \mathbb{N}}$ quasi–convex in $\mathbb{T}$?

Open Questions

For which non-torsion element $\beta \in \mathbb{T}$ does $\langle \beta \rangle$ contain a quasi-convex null sequence?
Let $S = \{b_n \alpha : \ n \in \mathbb{N}\} \subseteq \mathbb{T}$. Denote by τ_S the topology on $\mathbb{Z}$ of uniform convergence on the set S.
Is it correct that

$$(\mathbb{Z}, \tau_S)^\wedge = \langle \alpha \rangle \ ?$$

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Thank you for your attention.