A Dichotomy Theorem and Other Results for a Class of Quotients of Topological Groups

A. V. Arhangel'skii
MPGU and MGU, Moscow, RUSSIA

Suppose that G is a topological group and H is a closed subgroup of G. Then G/H stands for the quotient space of G which consists of left cosets xH, where $x \in G$. We call the spaces G/H so obtained coset spaces. They needn't be homeomorphic to a topological group, but are homogeneous and Tychonoff. The 2-dimensional Euclidean sphere S^2 is a coset space which is not homeomorphic to any topological group. (A space X is called homogeneous if for each pair x, y of points in X there exists a homeomorphism h of X onto itself such that $h(x) = y$). On the other hand, there exists a homogeneous compact Hausdorff space X such that X is not homeomorphic to any coset space [5]. A space X is said to be strongly locally homogeneous if for each $x \in X$ and every open neighbourhood U of x, there exists an open neighbourhood V of x such that $x \in V \subset U$ and, for every $z \in V$, there exists a homeomorphism h of X onto X such that $h(x) = z$ and $h(y) = y$, for each $y \in X \setminus V$.

It was proved by R.L. Ford in [3] that *if a zero-dimensional T_1-space X is homogeneous, then it is strongly locally homogeneous*. This fact was used to show that every homogeneous zero-dimensional compact Hausdorff space X can be represented as a coset space of a topological group (see Theorem 3.5.15 in [1][Theorem 3.5.15]). In particular, the two arrows compactum A_2 [4][3.10.C] is a coset space. However, A_2 is first-countable, compact, and non-metrizable. Therefore, A_2 is not dyadic. Recall in that every compact topological group is dyadic and every first-countable topological group is metrizable.

In this talk, coset spaces and remainders of coset spaces G/H are considered under the assumption that H is compact. “A space” always stands for “a Tychonoff topological space”. A remainder of a space X is the subspace $bX \setminus X$ of a compactification bX. Paracompact p-spaces are preimages of metrizable spaces under perfect mappings. A mapping is perfect if it is continuous, closed, and all fibers are compact. A Lindelöf p-space is a preimage of a separable metrizable space under a perfect mapping. Lindelöf Σ-spaces are continuous images of Lindelöf p-spaces. A space X is of point-countable type if each $x \in X$ is contained in a compact subspace F of X with a countable base of open neighbourhoods in X.

B.A. Efimov has shown that *every closed G_δ-subset of any compact topological group is a dyadic compactum*. M.M.Choban improved this result: *every compact G_δ-subset of a topological group is dyadic* [3]. Assume that $X = G/H$ is a coset space where the subgroup H is compact, and let F be a compact G_δ-subset of X. The natural mapping g of G onto $X = G/H$ is perfect, since H is compact. Therefore, the preimage of F under g is a compact G_δ-subset P of G. Since G is a topological group, it follows that P is dyadic. Hence, F is dyadic as well. Thus, the next theorem holds:

Theorem A

Suppose that G is a topological group, H is a compact subgroup of G, and F is a compact G_δ-subspace of the coset space G/H. Then F is a dyadic compactum.

Efimov's Theorem mentioned above cannot be extended to compact coset spaces: to see this, just take the two arrows compactum.

Theorem B

Suppose that G is a topological group, H is a compact subgroup of G, and U is an open subset of the coset space G/H such that $\overline{U}$ is compact. Then $\overline{U}$ is a dyadic compactum.

Another deep theorem on topological properties of topological groups was proved by M.G. Tkachenko: The Souslin number of any σ-compact group is countable. Later this theorem was extended by V.V. Uspenskiy to Lindelöf Σ-groups [1]. Below this result is extended to coset spaces with compact fibers.

Theorem C

Suppose that $X = G/H$ is a coset space such that the subgroup H is compact and X contains a dense Lindelöf Σ-subspace Z. Then the Souslin number of X is countable.

A similar result holds for the G_δ-cellularity.

The product of any family of pseudocompact topological groups is pseudocompact (Comfort and Ross). Below we use the following generalization of the theorem just mentioned:

Proposition D

If X is the topological product of a family $\{X_\alpha : \alpha \in A\}$ of pseudocompact topological spaces X_α such that X_α is an image of a topological group G_α under an open perfect mapping h_α, for each $\alpha \in A$. Then X is also pseudocompact.

Corollary E

If X is the topological product of a family $\{X_\alpha : \alpha \in A\}$ of pseudocompact coset spaces $X_\alpha = G_\alpha / H_\alpha$ where H_α is a compact subgroup of a topological group G_α, for each $\alpha \in A$. Then X is also pseudocompact.

It is consistent with ZFC that if a countable topological group G is a Fréchet-Urysohn space, then G is metrizable. Let us show that this theorem can be partially extended to coset spaces with compact fibers.

Theorem F

Suppose that $X = G/H$ is a coset space where the group G is countable, H is compact, and the space X is Fréchet-Urysohn. Then it is consistent with ZFC that X is metrizable.

Problem 1

Is it true that if a coset space G/H of a countable topological group G is a Fréchet-Urysohn space, then it is consistent that G/H is metrizable?

Problem 2

Suppose that G is a topological group with a countable network, and $X = G/H$ is a countable coset space where H is a compact subgroup of G. Then is it consistent with ZFC that X and G are metrizable?

Problem 3

Suppose that G is a topological group and $X = G/H$ is a countable coset space where H is a compact subgroup of G. Then is it consistent with ZFC that X is metrizable?

The next theorem extends a well-known result of B.A. Pasynkov on topological groups (see [1] for details) to arbitrary coset spaces with compact fibers.

Theorem F

If $X = G/H$ is a coset space where G is a topological group and H is a compact subgroup of G, and X contains a nonempty compact subspace with a countable base of open neighbourhoods in X, then X is a paracompact p-space.

Problem 4

Is every locally paracompact coset space G/H paracompact?

The answer to Problem 4 is positive when H is compact.

Theorem G

Suppose that G is a topological group and H is a compact subgroup of G such that the coset space G/H is locally paracompact (locally Čech-complete, locally Dieudonné complete). Then the coset space G/H is paracompact (Čech-complete, Dieudonné complete, respectively).

A space Y is called charming if it has a Lindelöf Σ-subspace Z such that $Y \setminus U$ is a Lindelöf Σ-space, for any open neighbourhood U of Z in Y [1]. Every charming space is Lindelöf. A space X is metric-friendly if there exists a σ-compact subspace Y of X such that $X \setminus U$ is a Lindelöf p-space, for every open neighbourhood U of Y in X, and the following two conditions are satisfied:
m_1) For every countable subset A of X, the closure of A in X is a Lindelöf p-space.
m_2) For every subset A of X such that $|A| \leq 2^\omega$, the closure of A in X is a Lindelöf Σ-space.
The next fact can be extracted from [1] and [2].

Theorem H

Every remainder of any paracompact p-space (in particular, any remainder of a metrizable space) is metric-friendly.

Proposition I

Suppose that f is a perfect mapping of a space X onto a space Y. Then X is metric-friendly if and only if Y is metric-friendly.

Problem 5

Suppose that G is a topological group, and let H be a compact subgroup of G. Then is it true that $dim(G/H) \leq dimG$? Is it true that $ind(G/H) \leq indG$?

It has been established in [5] that every remainder of any topological group is either pseudocompact or Lindelöf. This theorem is extended below to compactly-fibered coset spaces.

Proposition J

Suppose that X is a space such that either each remainder of X is Lindelöf, or each remainder of X is pseudocompact. Then every space Y which is an image of X under a perfect mapping also satisfies this condition: either each remainder of Y is Lindelöf, or each remainder of Y is pseudocompact.

Theorem K

Suppose that X is a compactly-fibered coset space, and $Y = bX \setminus X$ is a remainder of X in some compactification bX of X. Then the following conditions are equivalent:
1) Y is σ-metacompact;
2) Y is metacompact;
3)Y is paracompact;
4) Y is paralindelöf;
5) Y is Dieudonné complete;
6) Y is Hewitt-Nachbin-complete;
7) Y is Lindelöf;
8) Y is charming;
9) Y is metric-friendly.

The proof is based on the following fact:

Proposition L

Suppose that X is a compactly-fibered coset space with a Lindelöf remainder Y. Then Y is a metric-friendly space.

Thus, we have arrived at the following Dichotomy Theorem for compactly-fibered coset spaces:

Theorem M

For every compactly-fibered coset space X, either each remainder of X is metric-friendly, and X is a paracompact p-space, or every remainder of X is pseudocompact.

Theorem N

If the weight $w(X)$ of a compactly-fibered coset space X is not greater than 2^ω, then either each remainder Y of X is a Lindel´of Σ-space and X is a paracompact p-space, or every remainder of X is pseudocompact.

Corollary O

For every topological group G, either each remainder of G is metric-friendly and G is a paracompact p-space, or every remainder of G is pseudocompact.

Corollary P

If the weight $w(G)$ of a topological group is not greater than 2^ω, then either each remainder Y of G is a Lindelöf Σ-space and G is a paracompact p-space, or every remainder of G is pseudocompact.

A π-base for a space X at a subset F of X is a family γ of non-empty open subsets of X such that every open neighbourhood of F contains at least one element of γ. The next statement improves a result in [5].

Lemma CM

Suppose that G is a topological group with a non-empty compact subspace F of G such that G has a countable π-base at F. Then:
(i) There exists a compact subset P of the set FF^{-1} such that $e \in P$ and P has a countable base of open neighbourhoods in G.
(ii) Every remainder of G is a metric-friendly space, and G is a paracompact p-space.

Theorem R

Suppose that X is a compactly-fibered non-locally compact coset space with a remainder Y such that at least one of the following two conditions holds:
i_1) The π-character of the space Y is countable at each $y \in Y$, and the space Y is not countably compact;
i_2) The π-character of the space X (at some point of X) is countable.
Then X is metrizable, and Y is metric-friendly.

Proof.

Fix a topological group G, a compact subgroup H of G, and the quotient mapping $q : G \to G/H$ such that $X = G/H$. Then q is an open perfect mapping, and q can be extended to a perfect mapping $f : \beta G \to bX$, where bX is a compactification of X such that $Y = bX \setminus X$. Clearly, X and Y are nowhere locally compact. Therefore, X and Y are dense in bX.

Case 1. Assume that condition $i_1)$ holds. We will show that then $i_2)$ also holds.

Since Y is not countably compact, there exists an infinite countable discrete subspace A of Y which is closed in Y. Then A accumulates to some point $b \in X$. Clearly, bX has a countable π-base at each point of Y. Therefore, we can fix a countable π-base $\mathscr{P}_a$ at each $a \in A$. The family $\cup\{\mathscr{P}_a : a \in A\}$ is a countable π-base for bX at the point b. Taking into account that X is dense in bX, we conclude that there exists a countable π-base for X at b. Thus, condition $i_2)$ holds, and it is enough to consider this case:

Case 2. Condition i_2) holds.

The space X is homogeneous. Therefore, we can fix a countable π-base $\eta = \{V_n : n \in \omega\}$ for X at e. Since the map q is perfect, the family $\xi = \{q^{-1}(V_n) \cap G : n \in \omega\}$ is a countable π-base for G at the compact subset $q^{-1}(e)$ of G. But $q^{-1}(e)$ is the subgroup H of G. Therefore, by Lemma CM, there exists a compact subset P of HH^{-1} such that $e \in P$ and P has a countable base of open neighbourhoods in G. Using a standard obvious construction, we obtain a closed subgroup H_0 of G such that $H_0 \subset P$ and H_0 has a countable base of open neighbourhoods in G. Then we have: $H_0 \subset P \subset HH^{-1} = H$, that is, $H_0 \subset H$. The coset space G/H_0 is metrizable, since H_0 is compact and G/H_0 is first-countable (see [4] where it is shown that every first-countable compactly-fibered coset space is metrizable). Clearly, there is a natural continuous mapping s of G/H_0 onto G/H such that $q = sq_0$, where q_0 is the natural quotient mapping of G onto G/H_0. The mapping s is perfect, since q and q_0 are perfect. Therefore, the space $X = G/H$ is metrizable, since G/H_0 is metrizable. Hence, Y is metric-friendly. □

The above statement generalizes Kristensen's Theorem used in its proof.

Theorem S

Suppose that X is a compactly-fibered non-locally compact coset space with a remainder Y such that the space Y has a countable π-base (in itself). Then X is separable and metrizable, and Y is a Lindelöf p-space.

Theorem T

Suppose that $X = G/H$ is a compactly-fibered coset space with a compactification bX such that the tightness of bX is countable. Then X is metrizable.

In the above theorem, we cannot claim that X must be also separable. Indeed, an uncountable discrete topological group X can be represented as a dense subspace of an Eberlein compactum: just take the Alexandroff compactification of the discrete space X.

Theorem Q

Suppose that X is a compactly-fibered non-locally compact coset space with a remainder Y such that Y has a G_δ-diagonal. Then X and Y are separable and metrizable.

Proof.

Claim 1. Y is not countably compact.

Indeed, otherwise Y is metrizable and compact, by Chaber's Theorem [4]. This is a contradiction, since Y is not locally compact.

By the Dichotomy Theorem, either each remainder of X is charming and X is a paracompact p-space, or every remainder of X is pseudocompact.

Case 1. Y is charming and X is a paracompact p-space. Then Y has a countable network, since every charming space with a G_δ-diagonal does (see [1]). Therefore, the Souslin number of X is countable, since X and Y are both dense in bX. Since X is also a paracompact p-space, it follows that X is a Lindelöf p-space. Therefore, Y is a Lindelöf p-space, as it was shown in [4]. Since Y has a countable network, we conclude that Y has a countable base [2]. Now the metrization Theorem obtained above implies that X is metrizable. Hence, X is separable, since X is Lindelöf.
Case 2. Y is pseudocompact. Since Y is also a space with a G_δ-diagonal, it follows that Y is first-countable. By Claim 1, Y is not countably compact. Now it follows from the metrization Theorem above that X is metrizable. Hence, the remainder Y is charming [1]. Since Y is also pseudocompact, we conclude that Y is compact and hence, X is locally compact, a contradiction. Thus, case 2 is impossible, and therefore, X and Y are separable and metrizable. □

Theorem U

Suppose that X is a compactly-fibered non-locally compact coset space with a remainder Y such that Y has a point-countable base. Then X and Y are separable and metrizable.

Proof.

It is enough to consider the following two cases.
Case 1. Y is not countably compact. Then it X is metrizable and Y is metric-friendly. In particular, Y is Lindelöf. Since Y is also first-countable, it follows that $|Y| \leq 2^{\omega}$. Since Y is metric-friendly, we conclude that Y is a Lindelöf Σ-space. However, every Lindelöf Σ-space with a point-countable base has a countable base. Therefore, the Souslin number of X is countable. Hence X is separable, since X is metrizable. Thus, both X and Y are separable and metrizable.
Case 2. Y is countably compact. Then Y is a metrizable compactum, by a well-known Theorem of A.S. Mischenko [4]. We arrived at a contradiction. □

Theorem V

Suppose that X is a compactly-fibered non-locally compact coset space with a normal symmetrizable remainder Y. Then X and Y are separable and metrizable.

Proof.

Clearly, it is enough to consider the following two cases.
Case 1. Y is pseudocompact. Then Y is countably compact, since it is normal. Since Y is symmetrizable, it follows that Y is compact, by a theorem of S.J. Nedev [6]. Hence, X is locally compact, a contradiction. Thus, Case 1 is impossible.
Case 2. Y is Lindelöf. Then Y is hereditarily Lindelöf, by a theorem of Nedev [6]. Hence, Y is perfect, and the topological group X is separable and metrizable, by a theorem in [5]. Then Y is a Lindelöf p-space [3]. Since Y is symmetrizable, it follows that Y is separable and metrizable. □

Problem 6

Can the assumption that Y is normal be dropped in the last theorem?

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